Decomposition Methods for Complementarity Problems in Applied Economic Equilibrium Analysis

COPTA Research Talk
University of Wisconsin Madison

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Ann Arbor, MI

February 14, 2006

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Motivation

- For many years, from 1980 to 2000 we experienced ongoing improvements in computer software and hardware for mathematical programming, specifically in modeling languages and floating point speed.

- During this period integrated formulations were favored due their advantages in terms of clarity of ideas, compactness of model specification and ease of debugging.

- Complementarity methods have taken on increasing prominence because they provide a unifying approach to integrate of optimizing behavior in an equilibrium framework.

- Slowing rates of improvement in processors and modelling languages over the past five years have motivated a renewed interest in decomposition.

- In my opinion, effective decomposition algorithms require familiarity with how components of a given model interact.

- Optimization and complementarity "subproblems" can be specified and solved within modeling languages, eliminating the need for extensive programming.

Decomposition Frameworks

We are motivated by an interest in:

- Large-scale models
- Models in which agents or processes operate on inconsistent time scales
- Models in which non-convexities characterize certain elements of a model structure (I mention these models but do not go into details today.)

Algorithmic Approaches

- Sequential recalibration of multi-household demand by a single representative agent
- Sequential quadratic programming – approximation of general equilibrium demand system in a partial equilibrium subproblem.
- Sequential complementarity programming.

Template Applications

1. Large scale applications

   - Arrow-Debreu equilibrium models with many households.
   - Energy technology models which embed bottom-up and within top-down frameworks.

2. Models with components operating on inconsistent time scales

   - Integrated assessment modelling of climate change
   - Endogenous technical change through profit-oriented research and development

3. Models with non-convexities

   - Models of imperfect competition with segmented markets and heterogenous firms.

Newton/Josephy Method for Nonlinear Complementarity

Given: $F : R^n \to R^n$

Find $x \in R^n$ such that:

$$F(x) \perp x \geq 0$$

Repeat:

1. Construct an affine approximation:

$$L(x)|_{x=\bar{x}} = F(\bar{x}) + \nabla F(\bar{x})(x - \bar{x})$$

2. Solve

$$L(x) \perp x \geq 0$$

Josephy's Approach with NCP Subproblems

Given: $F : R^n \to R^n$

Find $x \in R^n$ such that:

$$F(x) \perp x \geq 0$$

Repeat:

1. Construct an approximation:

$$G(x)|_{x=\bar{x}} \approx F(x)|_{x \in B(\bar{x})}$$

2. Solve

$$G(x) \perp x \geq 0$$

Large Scale Application: Many Households

Representative agent model:

$$\max U(C) = \left(\sum_i \alpha_i C_i^{\rho} \right)^{1/\rho}$$

s.t.

$$\sum_i p_i C_i = M$$

Given $\bar{p}$, $\bar{C}$, calibrate share parameters given ρ:

$$\alpha_i = \lambda \bar{p}_i \bar{C}_i^{1-\rho}$$

Figure 1: The Benchmark Equilibrium

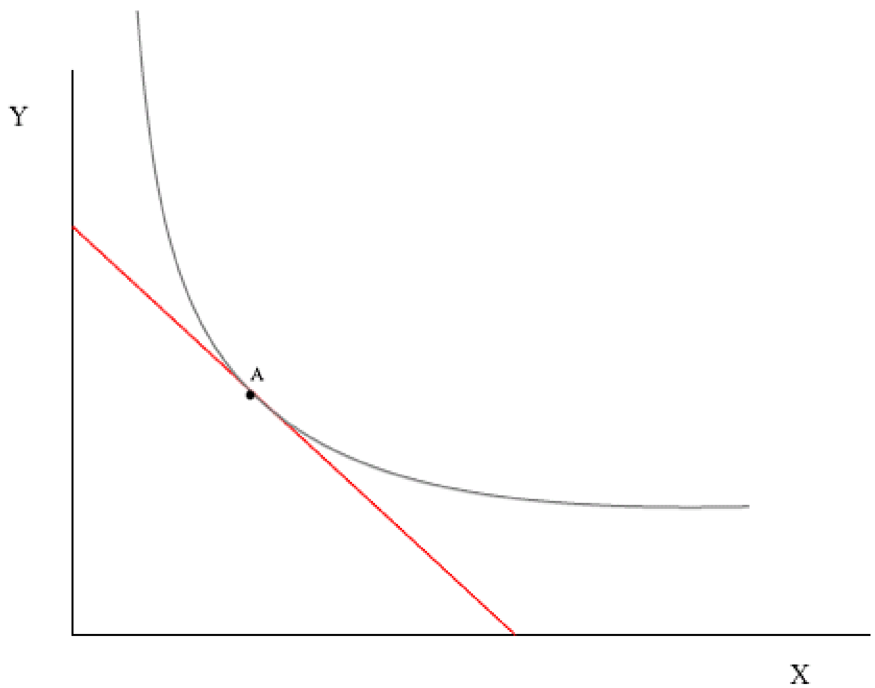
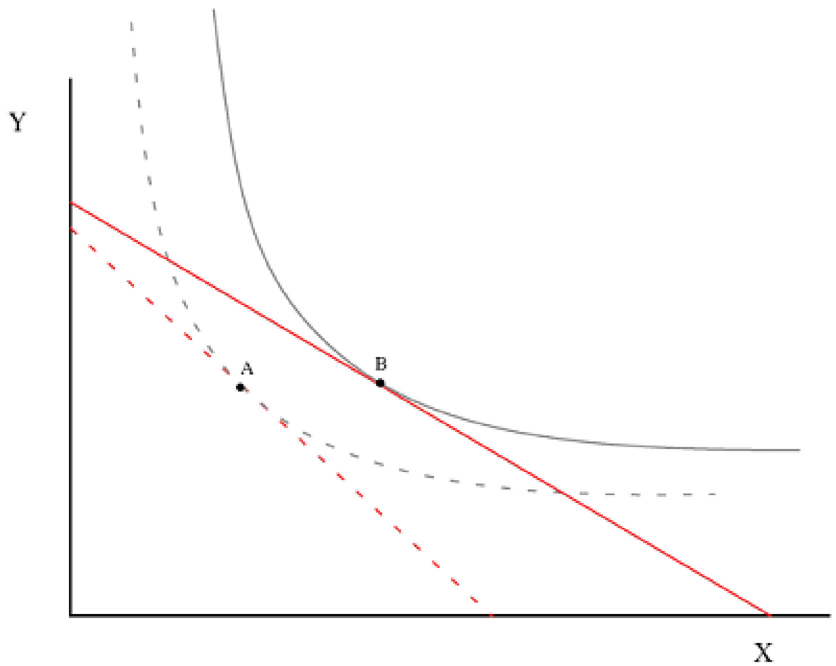


Figure 2: Solution to the Initial Representative Agent Model



Multiple household model:

$$\max u_h(c^h) = \left(\sum_i \alpha_i^h (c_i^h)^\rho \right)^{1/\rho}$$

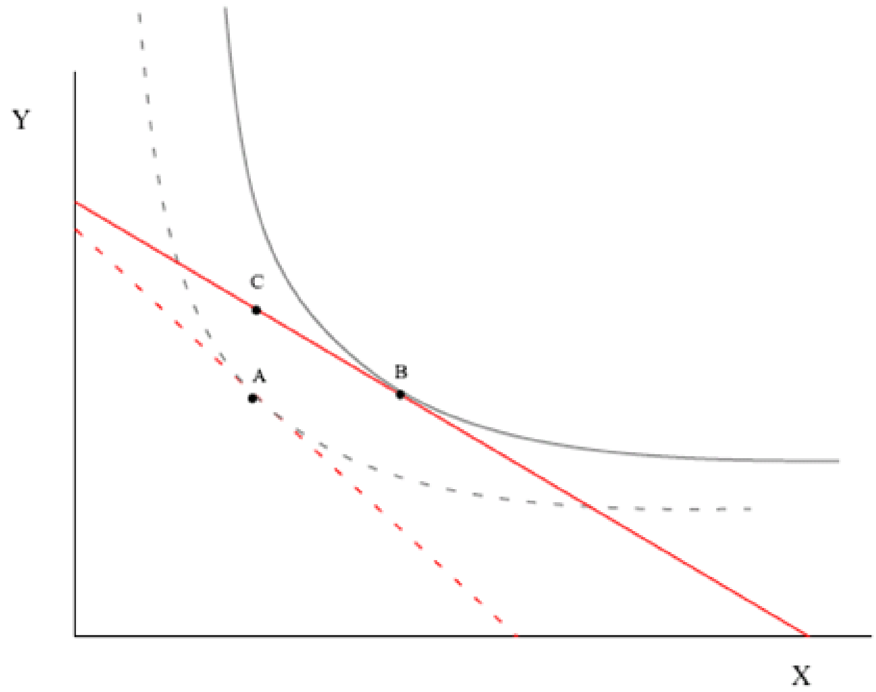
s.t.

$$\sum_i p_i c_i^h = M^h$$

Calibration based on consistent benchmark dataset:

$$\textstyle\sum_h \bar{c}_i^h = \bar{C}_i \text{ and } \alpha_i^h = \lambda \bar{p}_i (\bar{c}_i^h)^{1-\rho}$$

Figure 3: Evaluating Household Demands at New Prices

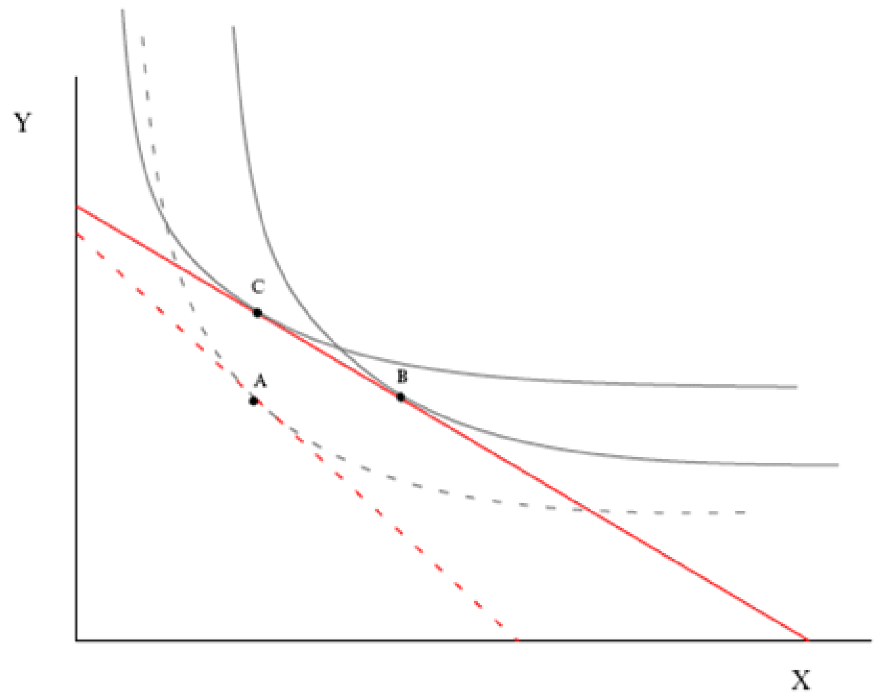


Recalibration step, taking:

$$\bar{C}_i^k = \sum_i c_{ih}(\bar{p}^k)$$

$$\alpha_i^k = \lambda \bar{p}_i^k \bar{(} C_i^k)^{1-\rho}$$

Figure 4: Recalibration of Preferences



Application: Aurbach-Kotlikoff Overlapping Generations Model

- Annual time steps over a 150 year horizon

- One generation leaves the economy and a new generation enters the economy in every period

- Economic lifetime of a single generation is 60 years (age 20 to age 80)

- Each cohort maximizes lifetime utility taking decisions of other agents as given.

- Original applications of this model relied on custom algorithms (Gauss-Seidel algorithms). New papers highlight advantages of complementarity format which accomodates corner solutions (e.g. retirement from the workforce).

Figure 6: Welfare Impacts

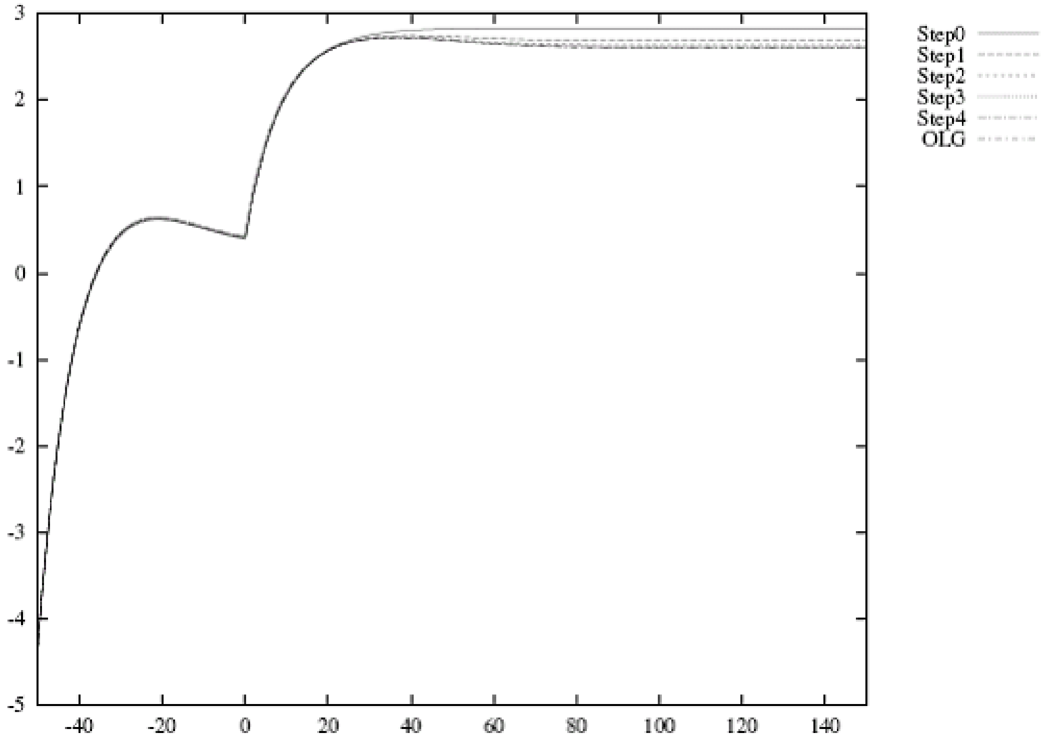
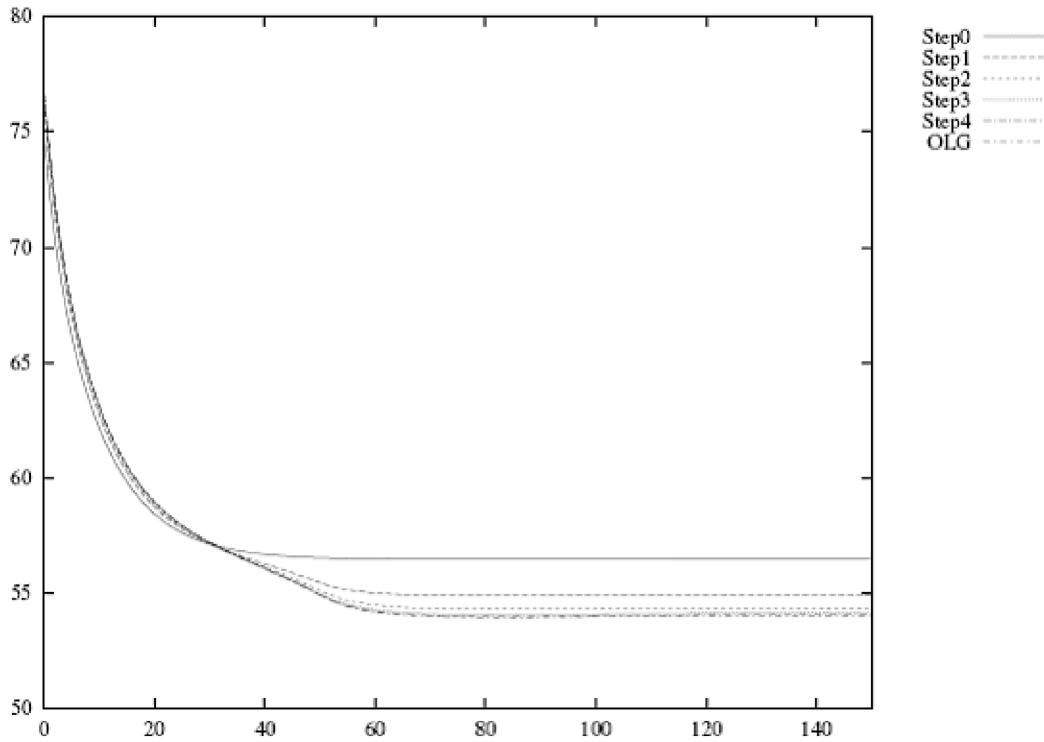


Figure 8: Investment Impacts



Convergence Theory?

Scarf (1960) provides a model which demonstrates the potential shortcomings of the sequential recalibration approach.

- n goods and n consumers

- Consumer i is endowed with one unit of good i and demands both goods i and $i+1$.

- Preferences are constant-elasticity-of-substitution:

$$U_i(d) = \left(\theta d_{ii}^{\rho} + (1-\theta) d_{ii+1}^{\rho}\right)^{1/\rho}$$

- Compare performance of the sequential recalibration algorithm with that of:

  1. Newton

  2. Tatonnement

  3. Sequential joint maximization (Negishi procedure)

Figure 9: Solution Process

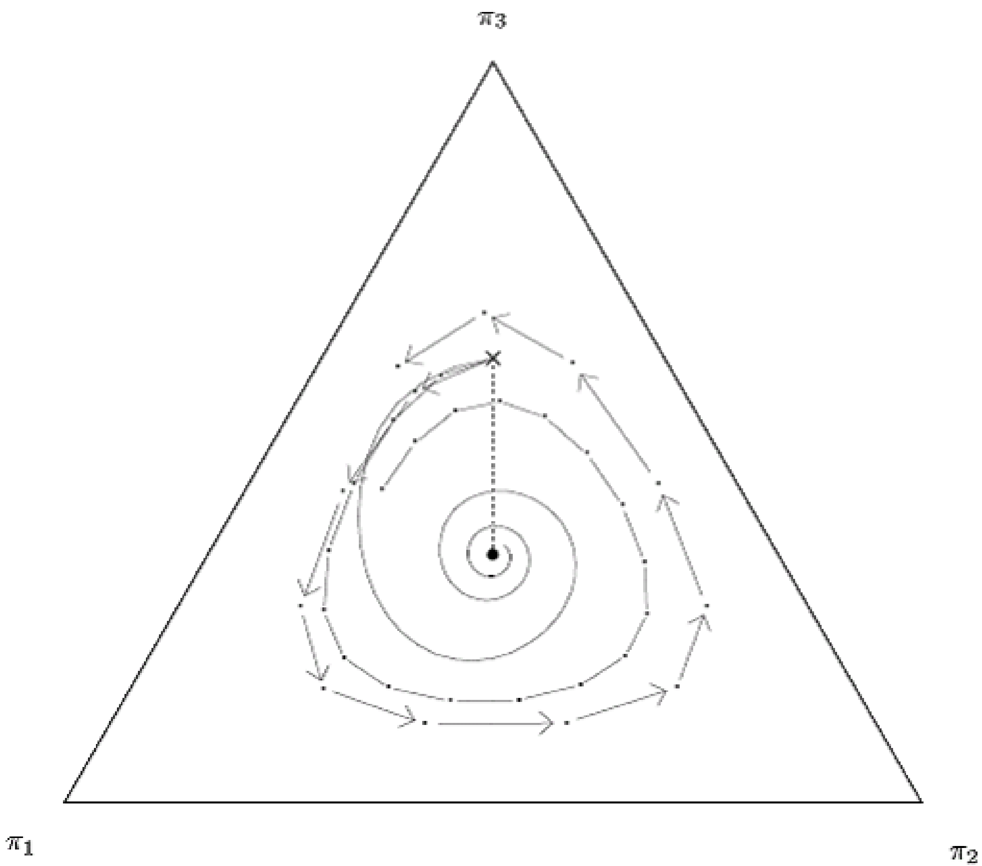
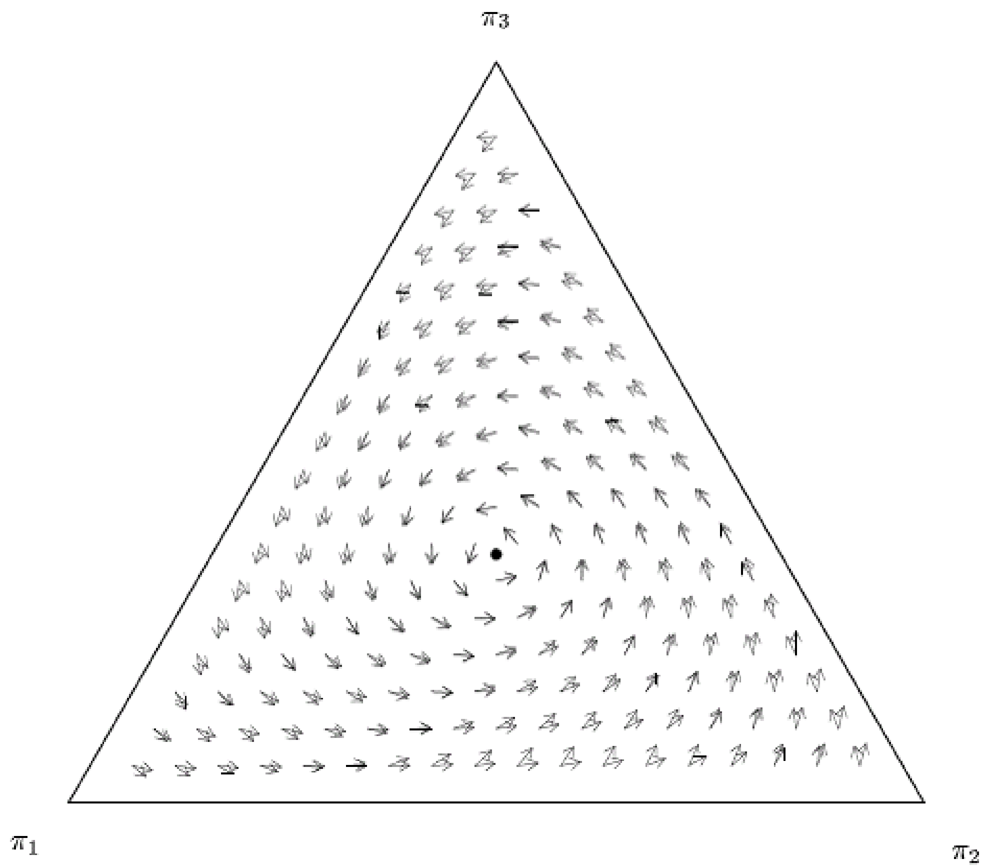


Figure 10: Comparison of SR and Tatonnement Fields



Large Scale Application: Many Technologies

The model: Top-Down Economic System, Bottom-Up Energy System

p denotes a non-negative n-vector in prices for all goods and factors,

y is a non-negative m-vector for activity levels of constant-returns-to-scale (CRTS) production sectors,

M is a h-vector of consumer income levels,

e represents a non-negative n-vector of net energy system outputs (including, for example, electricity, oil, coal, and natural gas supplies to residential, industrial, and commercial customers), and

x denotes a non-negative n-vector of energy system inputs (including labor, capital, and materials inputs).

Equilibrium

Zero profit:

$$-\Pi_j(p) \geq 0$$

Market clearance:

$$\sum_j \nabla \Pi_j(p)\ y_j + \sum_k \omega_k + e \geq \sum_k d_k(p, M_k) + x$$

Income balance:

$$M_k = p^T \left[\ \omega_k + \theta_k (e - x)\ \right]$$

Profit-maximizing energy sector:

e and x solve:

$$\max p^T(e-x)$$

subject to:

$$Ax + Bz \geq Ce$$

$$e, x \geq 0, \quad \ell \leq z \leq u$$

Attribution of energy sector rents:

$$M_k = p^T \omega_k + \Theta_k (\mu^T u + \lambda^T \ell) \tag{1}$$

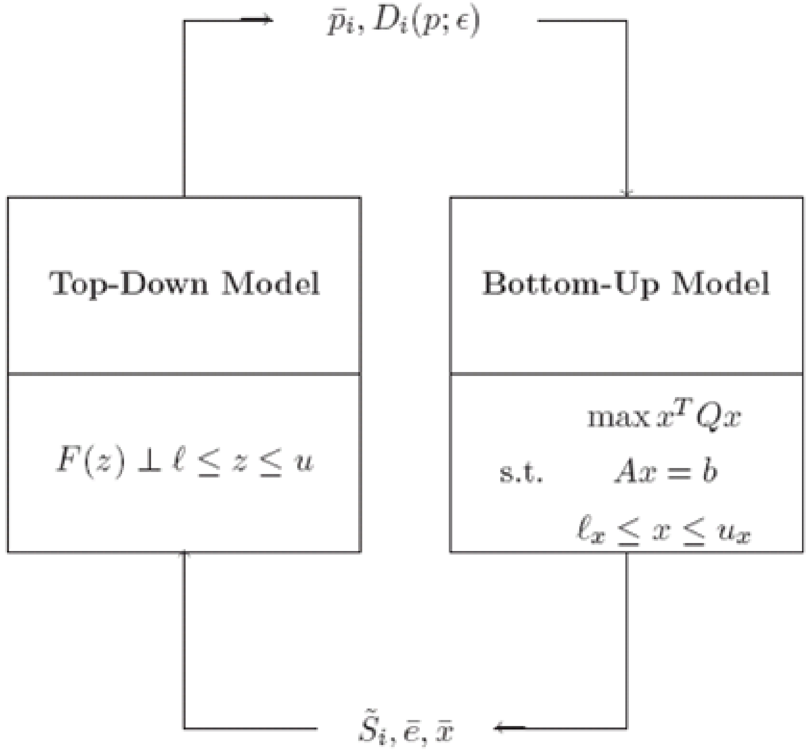
Model Dimensions

- m economic activities
- n energy goods
- M LP constraints
- N ancillary LP decision variables ($z \in R^N$)

Equation Count

- Integrated MCP model: $m + 3n + h + M + 3N$
- Economic model (without energy system): $m + n + h$
- LP energy model: M constraints and $N + 2n$ variables.

An Iterative Decomposition Algorithm



Constructing the Approximation

Demand for energy good i as:

$$e_i(p) = \bar{e}_i \left[1 - \epsilon_i (p_i/\bar{p}_i - 1)\right]$$

where ϵ_i is the elasticity of demand and $\bar{e}_i$ and $\bar{p}_i$ denote the observable reference quantities and prices for the demand function calibration.

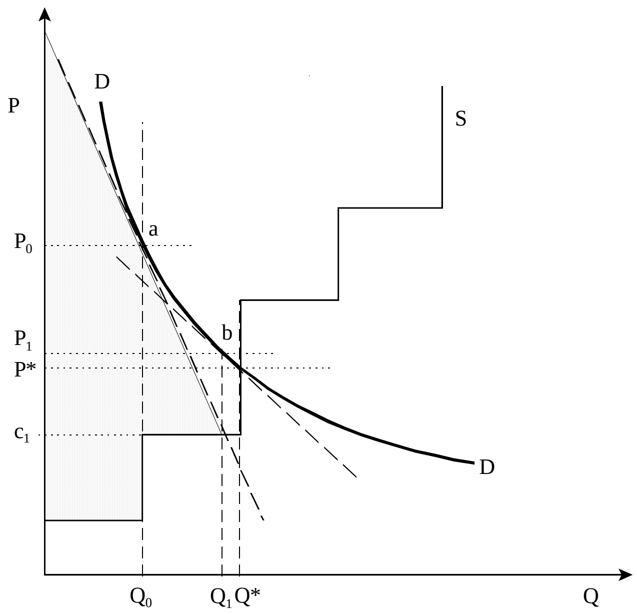
The calibrated inverse demand function is:

$$p_i(e) = \bar{p}_i \left[1 - (1 - e/\bar{e}_i)/\epsilon_i\right]$$

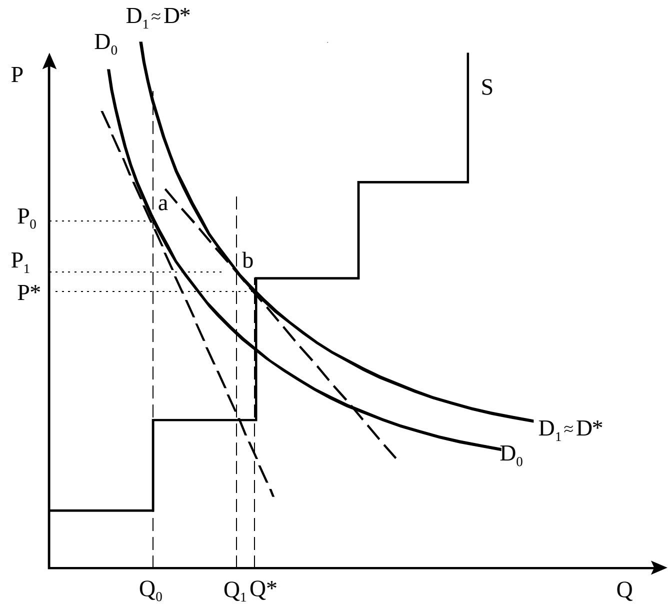
The *integrated* market demand function is:

$$\int p_i(e)de = \bar{p}_i e_i \left[1 - \frac{e_i - 2\bar{e}_i}{2\epsilon_i \bar{e}_i}\right],$$

Iterative Sequence: Single Market Partial Equilibrium



Iterative Sequence: Multimarket General Equilibriu



Decomposition to Deal with Ill-Conditioning

Integrated Assessment of Climate Change

- Broad classes of IA models: *policy simulation models* and *policy optimization models*

- Optimizing models are used for cost-benefit or cost-effectiveness analysis.

- IAMs must be solved over very long time horizons, as dictated by the climate component which operates over a period of 200 to 300 years.

- Existing IAMs are formulated as optimization models which are unable to address *second-best* phenomena (tax distortions, failures in the market for ideas, imperfect competition etc.)

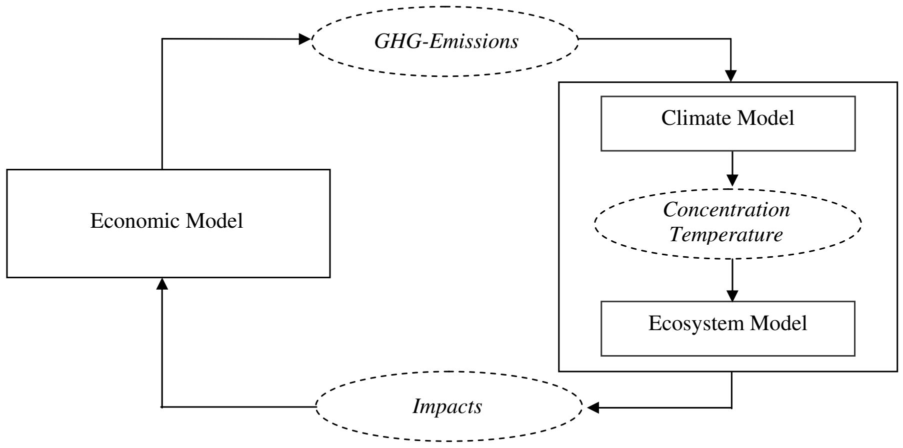
- Numerical problems are to be expected. Economic decisions are subject to *time preferences*, with discounting of future consumption. Goods valued at $1 today delivered one hundred years in the future are worth less than $0.01.

NLP Climate Policy Model

$$\max \sum_{t=0}^{\infty} \left(\frac{1}{1+\rho}\right)^t U(C_t, D_t)$$

$$\begin{aligned}
\text{s.t.} \qquad C_t &= F(K_t, D_t, E_t) - I_t \\
K_{t+1} &= (1-\delta)K_t + I_t \\
D_t &= D_t(T_t^E) \\
T_t^E &= H(S_t) \\
S_{t+1} &= G(S_t, E_t) \\
K_0 = \bar{K}_0, &\qquad S_0 = \bar{S}_0
\end{aligned}$$

Schematic Structure of Integrated Assessment Models for Climate Change



Linear Approximation of the Climate Model

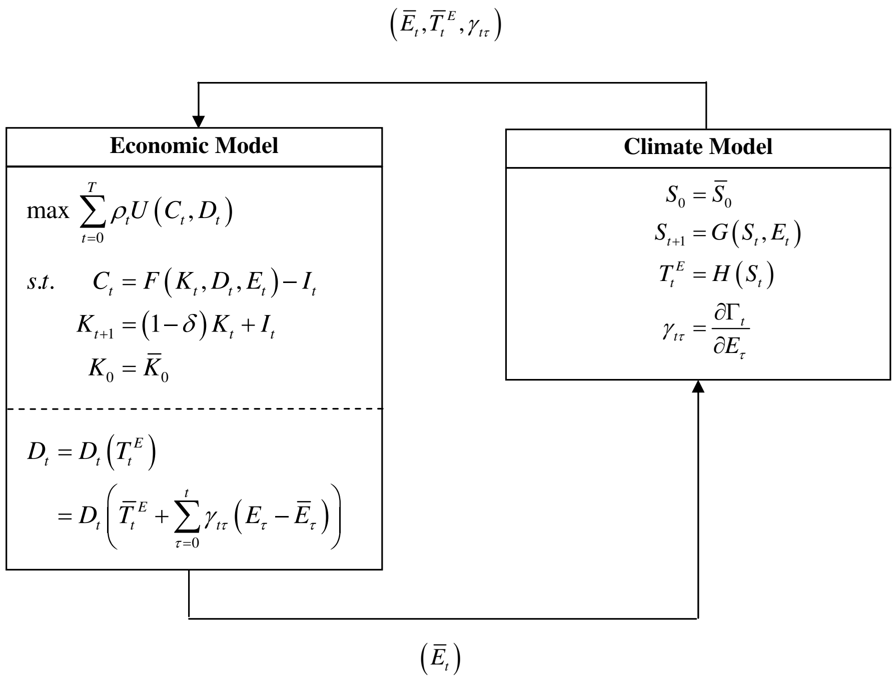
Merge $T_t^E = H(S_t)$ and $S_{t+1} = G(S_t, E_t)$ into a single equivalent equation

$$T_t^E = \Gamma_t(S_0, E_0, E_1, ..., E_{t-1})$$

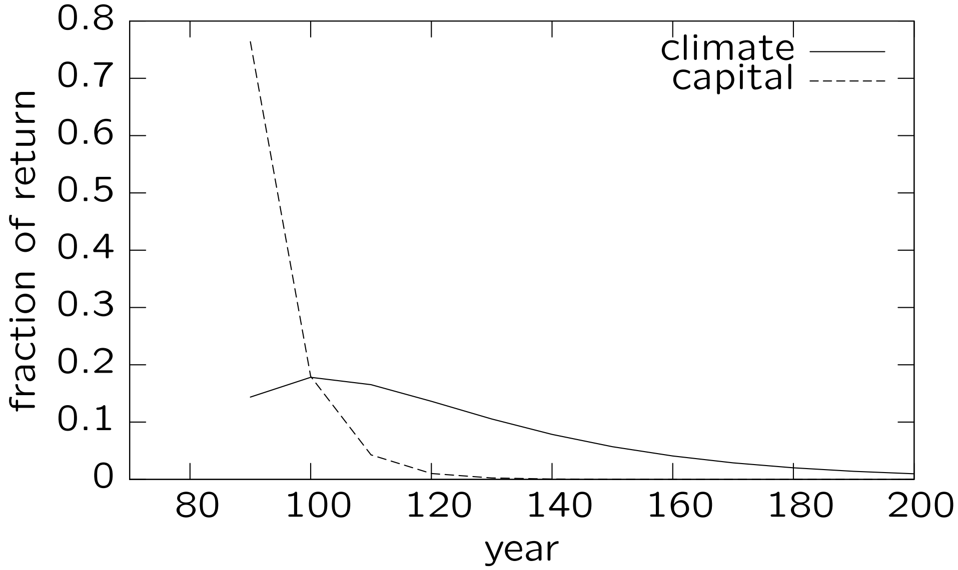
Associated first-order condition:

$$-p_t \frac{\partial F}{\partial E_t} = \sum_{\tau=t}^{\infty} \frac{\partial \Gamma_\tau}{\partial E_t} p_\tau^D = \sum_{\tau=t}^{T} \frac{\partial \Gamma_\tau}{\partial E_t} p_\tau^D + \sum_{\tau=T+1}^{\infty} \frac{\partial \Gamma_\tau}{\partial E_t} \tilde{p}_\tau^D$$

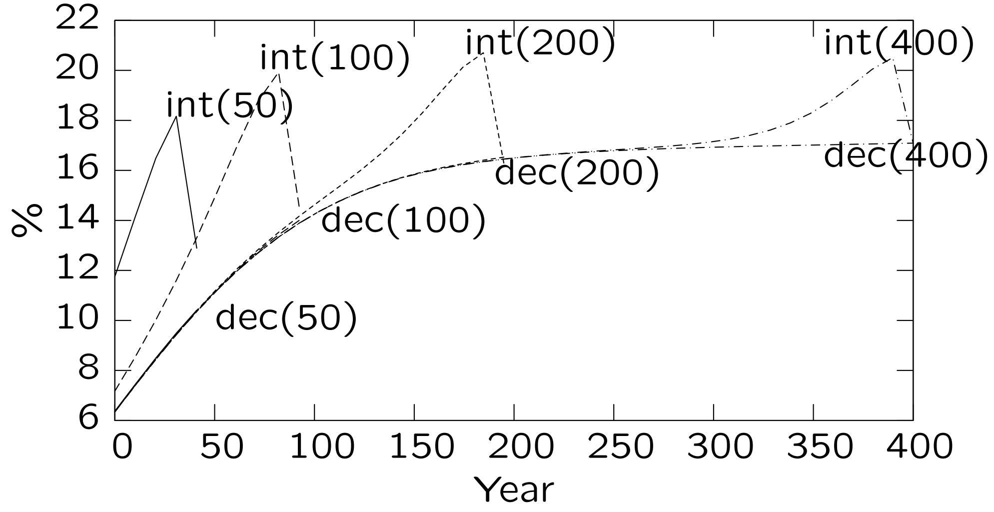
Logic of the Decomposition



Timing of Returns to Economic and Climate Investments



Sensitivity of the Emissions Control Rate



Conclusions

- Decomposition methods can be portrayed as an extension of the Josephy/Newton approach to a setting in which subproblems are nonlinear complementarity problems with approximations based on solution of related mathematical program(s).

- Decomposition methods can be effectively applied to large scale economic equilibrium problems

- Successive recalibration provides highly efficient techniques for Arrow-Debreu models with large numbers of consumers.

- Quadratic programming provides a effective scheme for integrating bottom-up linear programming submodels into a general equilibrium framework.

- Decomposition provides a means of interfacing models which operate on different time scales.

- The implementation of decomposition methods with a modeling language allows us to exploit model structure which would be undetectable within the solver.