Geometric Sums and Terminal Approximation of the Ramsey Model

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If we assume a constant consumption growth rate of γ from periods T to ∞, the Cobb-Douglas utility function can be written as:

$$\begin{aligned}
U &= \sum_{t=0}^{\infty} \left(\frac{1}{1+\rho}\right)^t \log(C_t) \\
&= \sum_{t=0}^{T-1} \left(\frac{1}{1+\rho}\right)^t \log(C_t) + \sum_{t=T}^{\infty} \left(\frac{1}{1+\rho}\right)^t \log\left(C_T(1+\gamma)^{t-T}\right) \\
&= \sum_{t=0}^{T-1} \left(\frac{1}{1+\rho}\right)^t \log(C_t) + \left(\frac{1}{1+\rho}\right)^T \left[\log(C_T) \sum_{\tau=0}^{\infty} \left(\frac{1}{1+\rho}\right)^\tau + \log(1+\gamma) \sum_{\tau=0}^{\infty} \tau \left(\frac{1}{1+\rho}\right)^\tau\right] \\
&= \sum_{t=0}^{T-1} \left(\frac{1}{1+\rho}\right)^t \log(C_t) + \left(\frac{1}{1+\rho}\right)^T \left[\log(C_T) \frac{1+\rho}{\rho} + \log(1+\gamma) \sum_{\tau=0}^{\infty} \tau \left(\frac{1}{1+\rho}\right)^\tau\right]
\end{aligned}$$

Where then can write:

$$U = \sum_{t=0}^{T} \beta_t \log(C_t) + \kappa$$

where

$$\beta_t = \begin{cases} \left(\frac{1}{1+\rho}\right)^t & t < T \\ \left(\frac{1}{1+\rho}\right)^T \frac{1}{\rho} & t = T \end{cases}$$

and

$$\kappa = \left(\frac{1}{1+\rho}\right)^T \sum_{\tau=0}^{\infty} \tau \left(\frac{1}{1+\rho}\right)^\tau$$

I prefer to take the steady-state growth and interest rates as model inputs in place of the discount factor. If the assumed steady-state growth rate is γ and the steady-state interest rate is r, you have a discount rate given by:

$$\rho = \frac{1+r}{1+\gamma} - 1 = \frac{r-\gamma}{1+r}$$

and

$$\beta_t = \begin{cases} \left(\frac{1+\gamma}{1+r}\right)^t & t < T \\ \left(\frac{1+\gamma}{1+r}\right)^T \frac{1+r}{r-\gamma} & t = T \end{cases}$$

If you are working with a CES utility function with an intertemporal elasticity equal to $1/\theta$, the algebra is a bit different. Using the calibrated share form and imposing the terminal assumption that $C_t = (1+g)^{t-T} C_T \quad \forall t \geq T$, we have:

$$
\begin{aligned}
U &= \left[\sum_{t=0}^{\infty} \left(\frac{1+\gamma}{1+r}\right)^t \left(\frac{C_t}{(1+\gamma)^t}\right)^{1-\theta} \right]^{1/1-\theta} \\
&= \left[\sum_{t=0}^{T-1} \left(\frac{1+\gamma}{1+r}\right)^t \left(\frac{C_t}{(1+\gamma)^t}\right)^{1-\theta} + \left(\frac{1+\gamma}{1+r}\right)^T \sum_{\tau=0}^{\infty} \left(\frac{1+\gamma}{1+r}\right)^\tau \left(\frac{C_T(1+\gamma)^\tau}{(1+\gamma)^\tau}\right)^{1-\theta} \right]^{1/1-\theta} \\
&= \left[\sum_{t=0}^{T-1} \left(\frac{1+\gamma}{1+r}\right)^t \left(\frac{C_t}{(1+\gamma)^t}\right)^{1-\theta} + \left(\frac{1+\gamma}{1+r}\right)^T C_T^{1-\theta} \sum_{\tau=0}^{\infty} \left(\frac{1+\gamma}{1+r}\right)^\tau \right]^{1/1-\theta} \\
&= \left[\sum_{t=0}^{T-1} \left(\frac{1+\gamma}{1+r}\right)^t \left(\frac{C_t}{(1+\gamma)^t}\right)^{1-\theta} + \frac{1+r}{r-\gamma} \left(\frac{1+\gamma}{1+r}\right)^T C_T^{1-\theta} \right]^{1/1-\theta} \\
&= \left(\sum_{t=0}^{T} \beta_t C_t^{1-\theta} \right)^{1/1-\theta}
\end{aligned}
$$

where

$$
\beta_t = \begin{cases} \left(\frac{(1+\gamma)^\theta}{1+r}\right)^t & t < T \\ \frac{1+r}{r-\gamma} \left(\frac{(1+\gamma)^\theta}{1+r}\right)^T & t = T \end{cases}
$$

Note that this expression corresponds precisely to the Cobb-Douglas result when we have $\theta = 1$.