Approximating Infinite-Horizon Models in a Complementarity Format: A Primer in Dynamic General Equilibrium Analysis

by

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Abstract

We demonstrate the usefulness of the complementarity format for approximating optimal saving and investment decisions in dynamic general equilibrium models. Our objective is in part pedagogic. The essential equations for alternative representations of the Ramsey model are presented in a compact and accessible format along with GAMS code as concrete illustration. We present a new method for approximating the infinite horizon equilibria with endogenous capital accumulation, and we demonstrate the advantages of this approach as compared with techniques originally developed for optimal planning models. The complementarity approach does not require an *ex ante* specification of the growth rate in the terminal period, and it is therefore suitable for models with endogenous growth or short time horizons. We also consider approximation issues arising in models with multiple infinitely-lived agents. In these models, changes in net indebtedness over a finite horizon must be estimated as part of the model in order to obtain a precise approximation with a small number of time periods.

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1. Introduction

Approximation of infinite-horizon models has a long-standing tradition in the economics literature. Most of this literature deals with optimization methods, whereas we demonstrate the usefulness of the complementarity format for approximating optimal saving and investment decisions in dynamic general equilibrium models. Our objective is pedagogic – the essential equations for a few models are presented in a compact and accessible format, along with computer programs which concretely illustrate the models. This approach is of interest to applied economists due to the availability of "off the shelf" software for processing these models (see Rutherford, [1995][1999a]).

There are two key issues involved in approximating an infinite horizon equilibrium for a neoclassical growth model: (i) what is the size of the capital stock in the terminal period?, and (ii) who owns the terminal capital stock? In this paper we demonstrate the advantages of the complementarity formulation for answering these questions compared with techniques originally developed for optimal planning models.

We begin the paper with the classical Ramsey analysis of optimal economic growth under certainty. This is a natural starting point because of the generic representation of financial markets. The model represents a closed economy with perfect competition in all markets, a representative consumer, and a constant rate of technological progress. Although the model is well studied in the economics literature (see, for example, Blanchard and Fischer [1989], and Barro and Sala-i-Martin [1995]), analytical methods have limitations. Numerical methods are, of course, always required for empirical analysis of policy issues, and they can provide helpful insights into properties of alternative formulations.

In section 2 we formulate the Ramsey model as a primal nonlinear program in quantities, as two different mixed complementarity problems (MCPs), and as a dual nonlinear program in prices. Preferences and technology are represented by utility and production functions in the primal formulation and by expenditure and cost functions in the dual model. The two MCP formulations can be interpreted as first-order necessary conditions for the nonlinear programming (NLP) models, and the complementarity problem associated with the dual nonlinear program is essentially Mathiesen's [1985] formulation of the Arrow-Debreu equilibrium model.

In section 3 we consider methods of approximating the infinite horizon. We present a new method for approximating the infinite horizon equilibria with endogenous capital accumulation, and we demonstrate the advantages of this approach as compared with techniques based on optimization methods. The complementarity approach does not require an *ex ante* specification of the growth rate in the terminal period, and it is therefore suitable for models with endogenous growth or short time horizons. We illustrate in a few examples that the complementarity formulation provides a more precise approximation of the infinite horizon equilibrium than optimization methods.

We also consider approximation issues arising in models with multiple infinitely-lived agents. In these models, changes in net indebtedness over a finite horizon must be calculated within the model in order to obtain a precise approximation with a small number of time periods. As illustration, in section 4 we present a Ramsey model with multiple regions, and we compare approximation errors for formulations with and without net changes in assets over the finite model horizon.

2. Four Formulations of the Single Sector Ramsey Model

A familiar representation of the Ramsey model of saving and investment begins with a single infinitely-lived representative agent. The closed economy consists of a household with an exogenous supply of labor over time. One good is produced in each period using inputs of labor and capital, and output in each period can be either consumed or invested. There is perfect competition in all markets and no taxes. Individuals are assumed to have an infinite horizon, and expectations by private agents are forward-looking and rational. Hence, all agents have perfect foresight because there is no uncertainty. These assumptions imply that the optimal allocation of resources by a central planner who maximizes the utility of the representative agent is identical to the optimal allocation of resources in an undistorted decentralized economy.

We present four alternative algebraic formulations of the Ramsey model, all of which produce an identical optimal allocation of resources given common assumptions regarding technology, preferences and initial endowments. Each formulation offers a different perspective into the workings of the Ramsey model. We begin with the most familiar format (primal NLP), and we proceed to two less familiar but convenient complementarity formats, and a dual optimization formulation. We feel that by laying out a set of mathematically-equivalent specifications, the researcher can develop basic insights into the nature of the equilibrium, which can be crucial when the time comes to interpret policy results from more complex models.

2.1. A Primal NLP Formulation

The primal NLP formulation is based on an explicit representation of the utility function for the single representative household. The social planner maximizes the present value of lifetime utility for the representative household:

$$U = \sum_{t=0}^{\infty} \left(\frac{1}{1+\rho} \right)^{t} u(C_t)$$

where ρ is the time preference rate, C_t is aggregate consumption in year t, and $u(\cdot)$ is the instantaneous utility of consumption.

The representative agent maximizes utility subject to the constraint that output in period t is either consumed or invested:

$$C_t + I_t = f(K_t)$$

where K_t is capital in period t, and I_t is investment in period t. Assuming strict monotonicity and concavity of the production function, we have that $f'(K_t) > 0$ and $f''(K_t) < 0$. It is convenient to think of the production function exhibiting constant returns to scale in capital and a second factor whose supply is exogenously specified. Labeling the second factor labor, we could, for example, represent diminishing returns to scale in capital through an underlying production function which exhibits constant returns to scale in labor and capital, i.e.

$$f(K_t) = F(K_t, \overline{L_t})$$

The capital stock in period *t* equals the capital stock at the start of the previous period less depreciation plus investment in the previous period. Hence, the capital stock is determined by

$$K_t = (1-\delta)K_{t-1} + I_{t-1}, \qquad K_0 = \overline{K_0}, \qquad I_t \geq 0$$

where δ is the annual rate of depreciation, and the initial capital stock in period $t=0$ is specified exogenously.

2.2. A Complementarity Formulation based on Karush-Kuhn-Tucker Conditions

It is a simple matter to pose a nonlinear program as a complementarity problem: just form the Lagrangian and differentiate. Introducing multipliers for aggregate output and capital stock, the above model produces the following system of first order conditions:

$$\left(\frac{1}{1+\rho}\right)^t \frac{\partial u(C_t)}{\partial C_t} = p_t$$

$$p_t^K = (1-\delta)\, p_{t+1}^K + p_t \frac{\partial f(K_t)}{\partial K_t}$$

$$p_t \geq p_{t+1}^K$$

$$C_t + I_t = f(K_t)$$

$$K_{t+1} = (1-\delta) K_t + I_t$$

$$K_0 = \overline{K_0}$$

$$I_t \geq 0, \quad I_t(p_t - p_{t+1}^K) = 0$$

where p_t is the output price in period t, and p_t^K is the price of capital in period t. As written, we take explicit account of the non-negativity constraint for investment and assume that all other variables are non-zero. Hence, we do not specify a set of complementarity relations for the other variables.

2.3. A Complementarity Formulation for Constant Returns Models

In order to exploit the complementarity format for economic equilibrium proposed by Mathiesen [1985], we expand the class of markets represented in the model in order to treat all production activities as constant returns to scale in model inputs. This is possible through the introduction of an additional primary factor, labor. We can then define the instantaneous unit cost function:

$$c(r_t, w_t) \equiv \min \; r_t^K K_t + w_t L_t \quad \text{s.t. } F(K_t, L_t) = 1$$

where r^K is the rental rate of capital, and w is the real wage rate. For example, if we assume that total factor productivity grows at a constant rate, γ, we have:

$$f(K_t) = (1+\gamma)^t \, K_t^\alpha$$

then:[1]

$$F(K_t, L_t) = K_t^\alpha \, \bar{L}_t^{1-\alpha} \quad \text{where} \quad \bar{L}_t = (1+\gamma)^{t/(1-\alpha)}$$

[1] That is, total factor productivity growth at rate γ requires Harrod-neutral labor productivity at rate $\hat{\gamma} = (1+\gamma)^{1/(1-\alpha)} - 1$ where α is the capital value share.

and:

$$c(r_t, w_t) = \left(\frac{r_t}{\alpha}\right)^{\alpha} \left(\frac{w_t}{1-\alpha}\right)^{1-\alpha}$$

This formulation further relies on the existence of closed-form demand functions which express consumption demands as a function of market prices and income, *M*. We then define:

$$D_t(\boldsymbol{p}, M) \equiv \underset{C_t}{\operatorname{argmax}} \sum_{t=0}^{\infty} \left(\frac{1}{1+\rho}\right)^{t} u(C_t) \quad \text{s.t.} \quad \sum_{t=0}^{\infty} p_t C_t = M$$

For example, if we have logarithmic instantaneous utility, we obtain the demand function:

$$D_t(p_t, M) = \frac{\rho}{(1+\rho)^{t+1}} \frac{M}{p_t}$$

or, if utility is isoelastic, $u(C_t) = \dfrac{C_t^{1-\theta}}{1-\theta}$, then we have:

$$D_t(\boldsymbol{p}, M) = \frac{M}{p_t^{1/\theta}} \frac{(1+\rho)^{-t/\theta}}{\sum_{t'} (1+\rho)^{-t'/\theta} \, p_{t'}^{(1-1/\theta)}}$$

Having defined uncompensated demand functions, we can characterize the equilibrium conditions in terms of three classes of equations: (i) zero profit conditions for all constant returns activities, (ii) market clearance conditions for all goods and factors, and (iii) income balance equations relating factor income to expenditure. The zero profit conditions for production, capital accumulation and investment are:

$$c(r_t, w_t) = p_t$$

$$p_t^K = (1-\delta)\, p_{t+1}^K + r_t$$

$$p_t \geq p_{t+1}^K$$

The market clearance conditions for capital stock, capital services, labor and output in each period are:

$$K_{t+1} = (1-\delta) K_t + I_t$$

$$K_t = Y_t \frac{\partial c(r_t, w_t)}{\partial r_t}$$

$$\overline{L}_t = Y_t \frac{\partial c(r_t, w_t)}{\partial w_t}$$

$$Y_t = D_t(\boldsymbol{p}, M) + I_t$$

An income-balance constraint relates the value of expenditure to factor earnings:

$$M = p_0^K\, \overline{K}_0 + \sum_{t=0}^{\infty} w_t \overline{L}_t$$

Due to homogeneity of cost and demand functions, the solution is not uniquely determined and the model determines only relative prices. A practical normalization in a model with one consumer is to fix *M=1* and omit the income constraint.

We assume non-negativity conditions for investment. Because of Walras' law and non-negativity of prices, complementary slackness conditions, $I_t\,(p_t - p^K_{t+1}) = 0$, arise as a feature of the definition of an equilibrium instead of an equilibrium condition *per se*.

2.4. A Dual NLP Formulation

In order to represent the model in dual form, it is necessary that the intertemporal utility function is linearly homogenous in consumption from period 0 to the infinite horizon. The restriction allows us to express indirect utility as the ratio of a function of market prices to the present value of income. Define the expenditure function:

$$e(\boldsymbol{p}) = \min \sum_{t=0}^{\infty} p_t C_t \qquad \text{s.t.} \qquad U(\boldsymbol{C}) = 1$$

It follows that the following nonlinear program has first order conditions which are equivalent to Mathiesen's complementarity formulation (for details, see Rutherford [1999b]):

$$\min \quad p_0^K K_0 + \sum_{t=0}^{\infty} w_t \overline{L_t} - M \log\left[e(\boldsymbol{p})\right]$$

subject to:

$$c(r_t, w_t) = p_t$$

$$p_t^K = (1-\delta)\, p_{t+1}^K + r_t^K$$

$$p_t \geq p_{t+1}^K$$

Observe that by Shephard's lemma:

$$\frac{\partial \log[e(\boldsymbol{p})]}{\partial p_t} = D_t(\boldsymbol{p}, M)$$

Associating Lagrange multipliers for the three classes of constraints with Y_t, K_t, and I_t, it can be seen that first order optimality conditions for the dual nonlinear program correspond to market clearance conditions in the complementarity model.

3. The Terminal Capital Stock

Numerical models can only be solved for a finite number of periods. Adjustments are therefore required to produce a model which approximates choices over the infinite horizon. In this section we propose a new method for approximating the infinite horizon equilibria with endogenous capital accumulation, and we demonstrate the advantages of this approach as compared with techniques originally developed for optimal planning models. The new method is only applicable in a complementarity format, but it may also be applied through sequential nonlinear programming. The advantage of the new approach is that it does not require an *ex ante* specification of the growth rate in the terminal period. It is therefore suitable for models with endogenous growth or short time horizons.

Barr and Manne [1967] introduced an early method for approximating the infinite horizon in optimal planning models which is still used in practice.[2] The method involves an increased weight on utility of consumption in the terminal period, and a constraint on investment in the terminal period. Assuming that the economy is in steady state by the terminal period T and growing at rate γ, the intertemporal utility function may be divided into two parts and written as:

[2] For other early papers, see Eckhaus and Parikh [1968], Chakravarty [1969], and Manne [1970].

$$U = \sum_{t=0}^{T-1} \left(\frac{1}{1+\rho} \right)^t \log(C_t) + \sum_{t=T}^{\infty} \left(\frac{1}{1+\rho} \right)^t \log\left(C_T (1+\gamma)^{(t-T)} \right)$$

$$= \sum_{t=0}^{T} \beta_t \log(C_t) + \mathit{Constant}$$

where we define the utility weight parameter in each period as:

$$\beta_t = \begin{cases} \left(\frac{1}{1+\rho} \right)^t & t < T \\ \left(\frac{1}{1+\rho} \right)^{T-1} \frac{1}{\rho} & t = T \end{cases}$$

and

$$\mathit{Constant} = \sum_{t=T}^{\infty} \left(\frac{1}{1+\rho} \right)^t \log(1+\gamma)^{t-T}$$

All quantities grow at the same rate in steady state. Gross investment in the terminal period is therefore determined by the size of the capital stock in the terminal period, the exogenous growth rate, and the capital depreciation rate. The constraint on investment in the terminal period assures sufficient investment to cover growth plus depreciation:

$$I_T = (\gamma + \delta) K_T$$

This approximation of the infinite horizon is *integrable* and can be applied in either NLP or MCP formulations of the Ramsey model. The limitation of this approach is that there is no easy way to determine whether the economy will be close to a steady-state in period *T*.[3]

Our new method for approximating the infinite time horizon relies on time-separable utility functions. We may then decompose the infinite horizon into two distinct optimization problems: one problem defined over the period *t=0* to *t=T*, and a second problem defined over the period *t=T+1* to infinity. In a single-sector model, the two sub-problems are linked through the capital stock in period *T+1*. The finite horizon problem for the representative household is:

$$\max \sum_{t=0}^{T} \left(\frac{1}{1+\rho} \right)^{t} \log(C_t)$$

subject to the intertemporal budget constraint:

$$\sum_{t=0}^{T} p_t C_t = \sum_{t=0}^{T} w_t L_t + p_0^K K_0 - p_{T+1}^K K_{T+1}$$

And the infinite horizon problem is:

$$\max \sum_{t=T+1}^{\infty} \left(\frac{1}{1+\rho} \right)^{t} \log(C_t)$$

subject to the intertemporal budget constraint:

$$\sum_{t=T+1}^{\infty} p_t C_t = \sum_{t=T+1}^{\infty} w_t L_t + p_{T+1}^K K_{T+1}$$

Having decomposed the model, a good terminal approximation is one in which the capital stock in period *T+1*, K_{T+1}, is close to the optimal value in the infinite-horizon program.

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[3] Appendix A provides four different Ramsey model versions formulated in GAMS (Brooke, Kendrick and Meeraus [1992], Rutherford [1995]).

If we know the "true" value of the capital stock in the post-terminal period then we can calculate the true consumption and saving paths during the transitional period. However, after a policy shock we do not know the "true" value of the capital stock in the post-terminal period. It could seem convenient to impose the long run steady-state value of the capital stock, but in that case the model horizon should be sufficiently long to converge to the steady state.

In a complementarity formulation we can include the post-terminal capital stock as an endogenous variable. As a system of equations, the extra variable requires a new equation. For this purpose we add an equation relating the growth rate of investment in the terminal period to the growth rate of output:[4]

$$I_T / I_{T-1} = Y_T / Y_{T-1}$$

We emphasize that this approach is appropriate only for complementarity models where the new constraint does not introduce a reduced cost for the variables appearing in the equation. For this reason, the termination method is not easily introduced in optimization models.[5]

The balanced investment growth constraint does not require that the model achieves the actual steady-state growth rate in period T. The advantage of the approach is that we do not have to impose a specific capital stock in the post-terminal period, nor a specific growth rate in the terminal period. This method is therefore suitable for models with endogenous growth where the terminal growth rate is not determined *ex ante*, like the model by Rutherford and Tarr [1999]. The termination method is illustrated in Appendix B using a Ramsey model formulated in GAMS/MPSGE (Rutherford [1999a]).

Figure 1 illustrates the terminal effect on investment for both terminal conditions using a

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[4] The use of aggregate output is not essential – investment growth could be related to consumption or any other "stable" quantity variable in the model.

[5] Application of this method in an optimal growth framework involves sequential optimization. Beginning with initial guess for the capital stock in the terminal period, we solve the primal NLP version of the model, and update the value of the capital stock in the terminal period with an iterative procedure using the constraint on the growth rate of investment in the terminal period. Specific programming details are provided in Appendix B.

single-sector Ramsey model.[6] In this calculation the initial capital stock is reduced by 20 percent to compare the two termination methods discussed above. We use a model simulation over a 100 year time horizon to represent the infinite-horizon saddle point path. We then compare computational results for the two terminal approximation methods by solving the model with each terminal condition for a 15 year time horizon. The model labeled "NLP" is based on the terminal condition by Barr and Manne, and the model labeled "MCP" is based on the state variable targeting procedure we propose. Deviations from the "true" saddle point path are smaller for the model based on state variable targeting (MCP) than for the model with increased weight on utility of consumption in the terminal period and a constraint on terminal investment (NLP). Several other variables exhibit deviations from related infinite horizon values, but investment is typically the most sensitive item.

The relationship between the "average" approximation error and the model horizon is illustrated in Figure 2. We define the average error as the weighted-sum of deviations from the "true" saddle point path for investment over the full model horizon. The weights are based on the present discounted value of future output in the initial steady state. Hence, the weights are determined by the interest rate and deviations from the "true" saddle point path for investment in the near future are weighted higher than similar deviations in the more distant future. Figure 2 shows that the average approximation error falls with the model horizon and is significantly smaller for the MCP model compared to the NLP model. The MCP model with state variable targeting can therefore obtain the same average precision with fewer periods than the NLP model with Barr and Manne's terminal constraint. For example, the MCP model with state variable targeting and a 10 year horizon produces almost the same average precision as an NLP model with Barr and Manne's terminal constraint and a 17 year horizon.

The results provide a practical argument for dynamic modeling in a complementarity format: the termination method is more precise. Hence, a given model can include fewer periods to approximate the infinite horizon saddle path when the state variable targeting method is used

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[6] The model is parameterized with a value share of capital equal to 0.36, the annual time preference rate is 5 percent, the annual steady state growth rate is 2 percent, and the annual rate of physical capital depreciation is 7 percent.

instead of Barr and Manne's terminal constraint.

4. Multiple Agents and Terminal Assets

Ownership of capital is an additional issue in dynamic models with multiple infinitely-lived agents. In these models, changes in net asset positions across households over a finite horizon must be calculated within the model in order to obtain a precise approximation with a small number of periods. We therefore have to distinguish between the value of capital goods and the net asset positions for private households.

To illustrate approximation issues arising in models with multiple infinitely-lived agents, we present and use a Ramsey model with multiple regions. Each region is endowed with an exogenous time path of labor and an initial capital stock, and the regions are linked through capital and consumption goods markets. Consumption goods are either consumed in the country of origin or used as intermediate inputs in other regions. Financial assets, on the other hand, are perfect substitutes and can move freely across regions, which implies that the interest rate is determined in the international financial market.

The intertemporal decision problem in each region is similar to the generic Ramsey model with a single household and can be decomposed into two distinct optimization problems: one problem defined over the period $t=0$ to $t=T$, and a second problem defined over the period $t=T+1$ to infinity. The representative household is concerned with the optimal distribution of consumption over time, and the two intertemporal sub-problems are thus linked via the stock of financial assets in period $T+1$. The finite horizon problem for the representative household in region r is:

$$\max \sum_{t=0}^{T} \left(\frac{1}{1+\rho} \right)^{t} \log\left(C_{r,t}\right)$$

subject to the intertemporal budget constraint:

$$\sum_{t=0}^{T} p_{r,t}\, C_{r,t} = \sum_{t=0}^{T} w_{r,t}\, L_{r,t} + A_{r,0} - A_{r,T+1}$$

And the infinite horizon problem is:

$$\max \sum_{t=T+1}^{\infty} \left(\frac{1}{1+\rho}\right)^{t} \log\left(C_{r,t}\right)$$

subject to the intertemporal budget constraint:

$$\sum_{t=T+1}^{\infty} p_{r,t}\, C_{r,t} = \sum_{t=T+1}^{\infty} w_{r,t}\, L_{r,t} + A_{r,T+1}$$

where $A_{r,t}$ is the stock of financial assets in region r in period t.

Having decomposed the intertemporal decision problem, a good terminal approximation is one in which the net asset position in each region in period $T+1$, $A_{r,T+1}$, is close to the optimal value in the infinite-horizon program. Our starting point is the same as before. We exploit the complementarity format and apply the state variable targeting procedure to determine the post-terminal capital stock in every region, $K_{r,T+1}$. Hence, we include the post-terminal capital stock in every region as endogenous variables and add an equation for each capital stock that relates the growth rate of investment in the terminal period to the growth rate of output in the given region:

$$I_{r,T} / I_{r,T-1} = Y_{r,T} / Y_{r,T-1}$$

The stock of financial assets in a given region may be different from the value of the capital stock in that region. We therefore have to adjust the intertemporal budget constraint over the finite horizon to account for changes in net financial wealth. Having determined the post-terminal capital stock in every region, the intertemporal budget constraint for a given region is adjusted by the difference between the region's ownershare of global financial assets and the value of the region's capital stock in period *T+1*:

$$\Delta_r = \theta_r \sum_s p^K_{s,T+1} K_{s,T+1} - p^K_{r,T+1} K_{r,T+1}$$

where θ_r is the ownershare of global financial assets by region r in period *T+1*.

We know from the intertemporal budget constraint that the stock of financial assets in period *T+1* is equal to the difference between the present value of consumption expenditures and labor earnings from period *T+1* to infinity. All quantities in a specific region grow at the same rate in steady state, γ_r, and the ownershares of global financial assets across regions in period *T+1* can be determined by:

$$\theta_r = \frac{\sum_{t=T+1}^{\infty} \left(p_{r,T} C_{r,T} - w_{r,T} L_{r,T}\right)\left((1+\gamma_r)/(1+r)\right)^{(t-T)}}{\sum_s \sum_{t=T+1}^{\infty} \left(p_{s,T} C_{s,T} - w_{s,T} L_{s,T}\right)\left((1+\gamma_s)/(1+r)\right)^{(t-T)}}$$

The expression is simplified somewhat if all regions are on a common growth path at the end of the model horizon. In this case, the ownershares of global financial assets across regions can be determined by the difference between consumption expenditures and labor earnings in period *T*:

$$\theta_r = \left(\frac{p_{r,T} C_{r,T} - w_{r,T} L_{r,T}}{\sum_s (p_{s,T} C_{s,T} - w_{s,T} L_{s,T})} \right)$$

Figure 3 illustrates the effects on capital flows for a model with and without adjustment for changes in net indebtedness over the finite horizon. We apply a two-region Ramsey model, and the initial capital stock is reduced by 20 percent in one region.[7] A model simulation over a 100 year time horizon represents the infinite-horizon saddle point path, and we compare computational results for the two terminal approximation methods by solving the model with each intertemporal budget constraint for a 15 year time horizon. The model labeled "Capital adjustment" is based on the adjusted intertemporal budget constraint over the finite time horizon, and the model labeled "Original budget constraint" is based on the original intertemporal budget constraint. Figure 3 illustrates that deviations from the "true" saddle point path are smaller when the intertemporal budget constraint is adjusted for changes in net indebtedness over the finite time horizon compared to a model version without the adjustment. Hence, changes in net asset positions across households over a finite horizon must be calculated within the model in order to obtain a precise approximation with a small number of periods.

5. Conclusion

We have proposed a new method for approximating the infinite horizon equilibrium with endogenous capital formation and demonstrated the usefulness of the complementarity format for determining optimal saving and investment decisions in dynamic general equilibrium models. The state variable targeting method has two advantages compared with techniques based on optimization methods. First, state variable targeting provides a more precise approximation of the infinite horizon equilibria than optimization methods. The results thus provide a practical argument for dynamic modeling in a complementarity format, because a given model can include fewer periods to approximate the infinite horizon saddle path when the state variable targeting method is used. Second, state variable targeting does not require an *ex ante* specification of the growth rate in the terminal period. This method is therefore suitable for neoclassical models with endogenous growth where the growth rate in the terminal period is not determined *ex ante*.

[7] Mention here the basic parameter values in the model.

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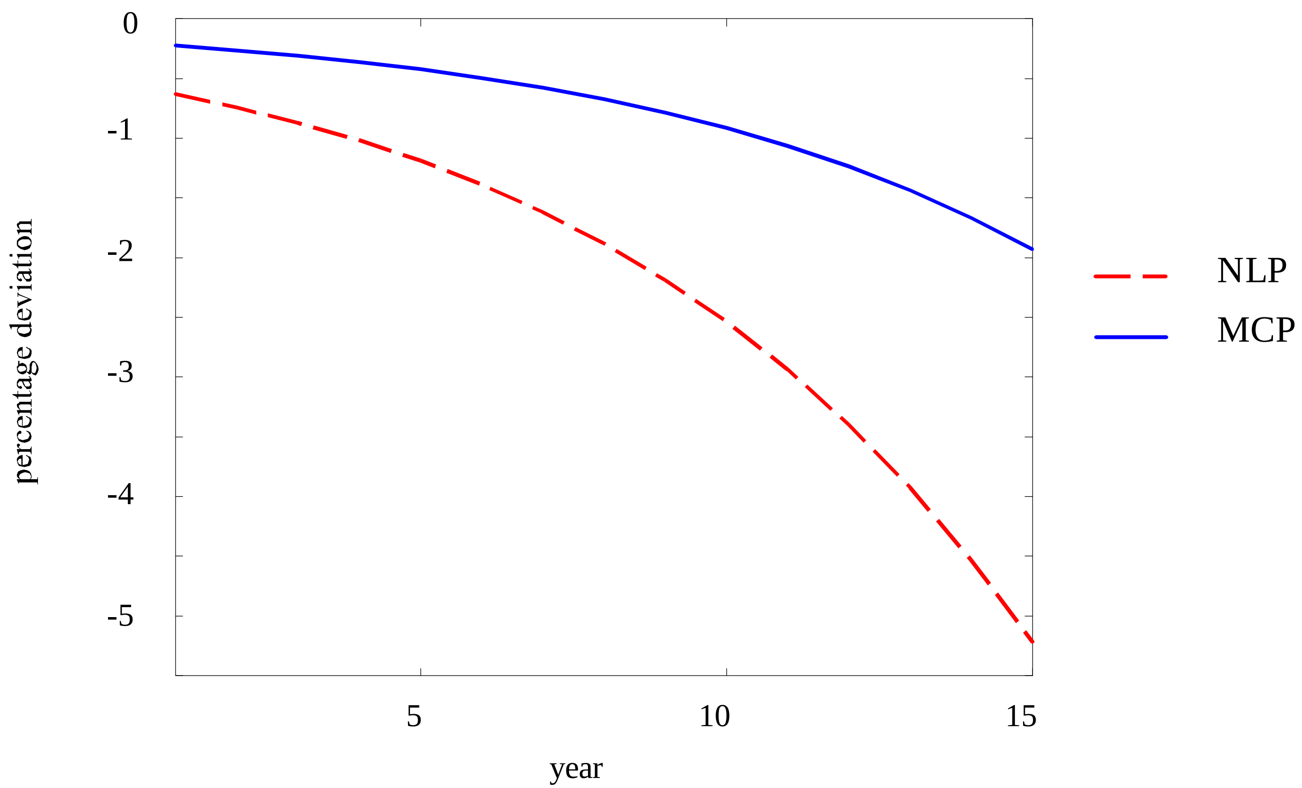

Figure 1. Approximation Errors in Investment (15 Year Time Horizon). The initial capital stock is reduced by 20 percent, and the model is solved with each terminal condition for a 15 year time horizon. The model labeled "NLP" is based on the terminal condition by Barr and Manne, and the model labeled "MCP" is based on the state variable targeting procedure we propose. Deviations from the "true" saddle point path are smaller for the model based on state variable targeting (MCP) than for the model with increased weight on utility of consumption in the terminal period and a constraint on terminal investment (NLP).

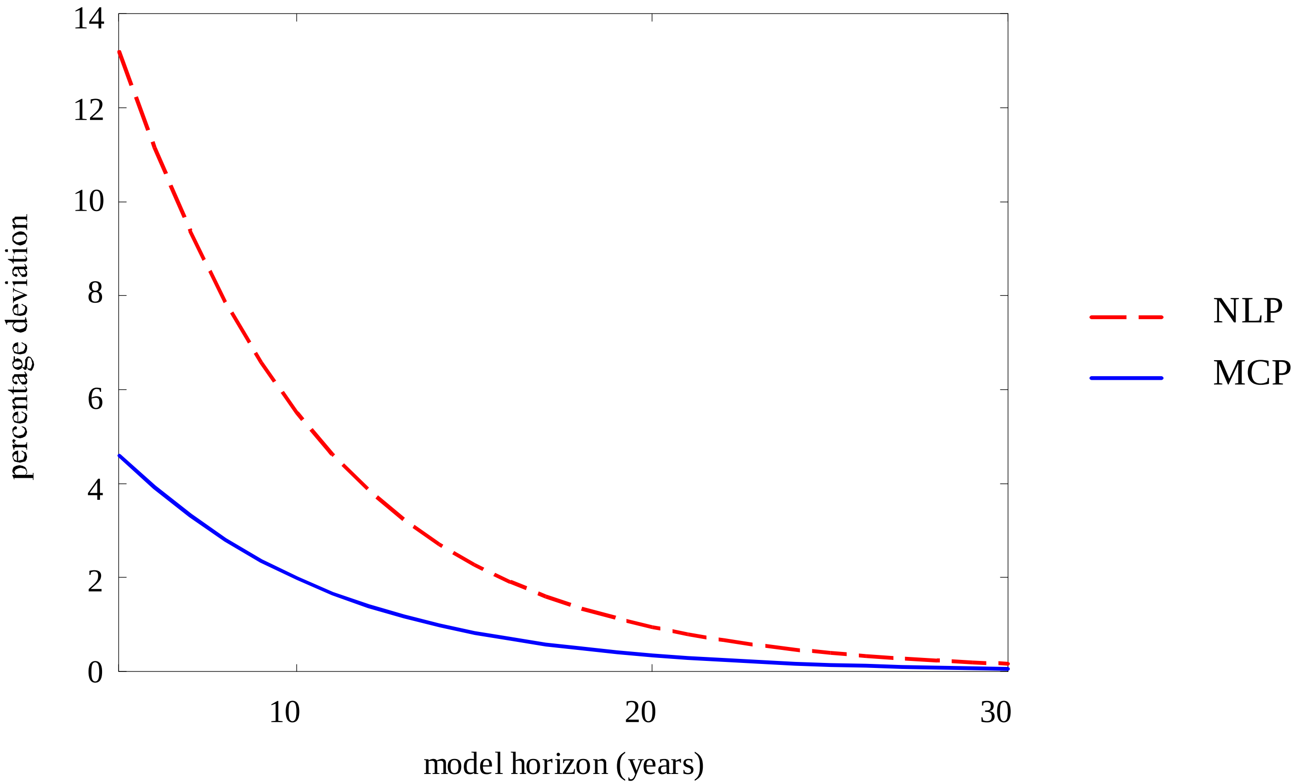

Figure 2. Discounted Average Approximation Error. The average error is defined as the weighted-sum of deviations from the "true" saddle point path for investment over the full model horizon. The weights are determined by the interest rate and deviations from the "true" saddle point path for investment in the near future are weighted higher than similar deviations in the more distant future. The average approximation error falls with the model horizon and is significantly smaller for the MCP model compared to the NLP model. The MCP model can therefore obtain the same average precision with fewer periods than the NLP model.

Appendix A. GAMS Code for Alternative Formulations of the Ramsey Model

```
$TITLE Basic Data.

SET     T /1*20/
SET     TFIRST(T), TLAST(T);
TFIRST(T) = YES$(ORD(T) EQ 1);
TLAST(T)  = YES$(ORD(T) EQ CARD(T));

SCALAR
        G       Growth rate                             /0.02/,
        IR      Interest rate                           /0.05/,
        K0      Capital-output ratio                    /3.00/,
        DELTA   Depreciation rate                       /0.07/,
        Kstock  Capital stock index                     /1.00/,
        I0      Base year investment,
        L0      Base year labor input,
        C0      Base year consumption,
        KVS     Base year capital value share;

I0 = (DELTA + G) * K0;
L0 = 1 - K0 * (DELTA + IR);
C0 = 1 - (DELTA + G) * K0;
KVS = K0 * (DELTA + IR);

PARAMETER       QREF(T)         Reference quantity path,
                PREF(T)         Reference price path;

QREF(T) = (1+G)**(ORD(T)-1);
PREF(T) = (1/(1+IR))**(ORD(T)-1);

PARAMETER ALPHA;
ALPHA(T) = ((1+G)/(1+IR))**(ORD(T)-1);
ALPHA(TLAST) = ALPHA(TLAST) / (1-(1+G)/(1+IR));


$TITLE Ramsey Model: Barr-Manne Primal.

$SYSINCLUDE GAMS-F
$INCLUDE DATA

*       Declare the production and utility functions here:

F(K) == (K/K0)**KVS * QREF(T)**(1-KVS);
U(C) == LOG(C);

POSITIVE VARIABLES
        K(T)            Capital stock
        C(T)            Consumption
        I(T)            Investment;

VARIABLES
        UTILITY         Utility function;

EQUATIONS
        CC(T)           Capacity constraint,
        KK(T)           Capital balance,
        TC(T)           Terminal condition (provides for post-terminal growth),
        UTIL            Discounted log of consumption: objective function;

CC(T)..         F(K(T)) =E= C(T) + I(T);

KK(T+1)..       (1-DELTA)*K(T) + I(T) =G= K(T+1);

TC(TLAST)..     I(TLAST) =G= (G+DELTA)*K(TLAST);

UTIL..          UTILITY =E= -SUM(T, ALPHA(T) * U(C(T)));

MODEL PRIMAL /CC, KK, TC, UTIL/;
```

```
C.L(T)       = C0;
K.LO(T)      = 1E-4;
C.LO(T)      = 1E-4;
I.LO(T)      = 1E-4;
K.FX(TFIRST) = K0;

SOLVE PRIMAL MINIMIZING UTILITY USING NLP;

K.FX(TFIRST) = K0*0.80;
SOLVE PRIMAL MINIMIZING UTILITY USING NLP;


$TITLE Ramsey Model: Barr-Manne KKT.

$INCLUDE DATA

POSITIVE VARIABLES
       P(T)           Price of output,
       PK(T)          Price of capital,
       RK(T)          Price of rental capital,
       PTC(T)         Price of capital in terminal period
       K(T)           Capital stock
       C(T)           Consumption
       I(T)           Investment;

EQUATIONS
       CC(T)          Capacity constraint,
       KK(T)          Capital balance,
       TC(T)          Terminal condition (provides for post-terminal growth),
       OPT_C(T)       First order optimality condition for C,
       OPT_K(T)       First order optimality condition for K,
       OPT_I(T)       First order optimality condition for I;

CC(T)..        (K(T)/K0)**KVS * QREF(T)**(1-KVS) =E= C(T) + I(T);

KK(T)..        (1-DELTA)*K(T-1) + (K0*KSTOCK)$TFIRST(T) + I(T-1) =G= K(T);

TC(TLAST)..    I(TLAST) =G= (G+DELTA)*K(TLAST);

OPT_C(T)..     C(T)*P(T) =E= C0 * ALPHA(T);

OPT_K(T)..     PK(T) + (PTC(T)*(G+DELTA))$TLAST(T) =E=
                       P(T)*KVS*(K(T)/K0)**KVS * QREF(T)**(1-KVS)/K(T) + PK(T+1)*(1-DELTA);

OPT_I(T)..     P(T) =E= PK(T+1) + PTC(T)$TLAST(T);

MODEL KKT / OPT_C.C, OPT_K.K, OPT_I.I, CC.P, KK.PK, TC.PTC /;

C.L(T)   = C0 * QREF(T);
I.L(T)   = I0 * QREF(T);
K.L(T)   = K0 * QREF(T);

P.L(T)   = PREF(T);
PK.L(T)  = PREF(T) * (1+IR);
PTC.L(T) = PREF(T);

PK.LO(T) = 1E-6;
PK.UP(T) = +INF;

SOLVE KKT USING MCP;

KSTOCK = 0.8;
SOLVE KKT USING MCP;


$TITLE Ramsey Model: Barr-Manne MCP.

$SYSINCLUDE GAMS-F
$INCLUDE DATA

$if not setglobal termcnd $setglobal termcnd NLP
```

```
SCALAR NLPTERM /0/, MCPTERM/0/;
%termcnd%term = 1;

*       Default data set up for NLP termination:

IF (mcpterm, ALPHA(T) = ((1+G)/(1+IR))**(ORD(T)-1));

ALIAS (T,TT);
ALPHA(T) = ALPHA(T) / SUM(TT, ALPHA(TT));

*       Declare the production and utility functions here:

F(RK,PL) == (RK/(IR+DELTA))**KVS * PL**(1-KVS);
U(P) == PROD(T, (P(T)/PREF(T))**ALPHA(T));

POSITIVE VARIABLES
        Y(T)            Output
        I(T)            Investment
        K(T)            Capital stock
        P(T)            Output price
        RK(T)           Return to capital
        U               Utility
        PU              Unit expenditure function
        PK(T)           Capital price
        PL(T)           Wage rate
        PKT             Terminal capital
        RA              Representative agent
        TK              Post-terminal capital stock;

EQUATIONS
        PR_Y(T)         Zero profit condition for output,
        PR_C(T)         Zero profit condition for consumption,
        PR_K(T)         Zero profit condition for capital,
        PR_I(T)         Zero profit condition for investment,
        PR_U            Zero profit condition for utility,

        M_P(T)          Market clearing for output,
        M_PK(T)         Market clearing for capital,
        M_RK(T)         Market clearing for rental capital,
        M_PL(T)         Market clearing for labor,
        M_PU            Market clearing for utility,
        M_PKT           Market clearing for terminal investment,

        I_RA            Income balance for representative agent,
        TERMK           Terminal constraint for capital stock;

PR_Y(T)..       F(RK(T),PL(T)) =E= P(T);

PR_U..          U(P) =E= PU;

PR_K(T)..       PK(T) + ( (PKT*(G+DELTA))$TLAST(T) )$NLPTERM =E=
                        RK(T) + (1-DELTA)*(PK(T+1) + (PKT$TLAST(T))$MCPTERM);

PR_I(T)..       P(T) =E= PK(T+1) + PKT$TLAST(T);

M_P(T)..        Y(T) =E= ALPHA(T) * PU * U / P(T) + I(T);

M_PU..          U * PU =E= RA;

M_PK(T)..       K(T) =E= (1-DELTA)*K(T-1) + I(T-1) + (K0*KSTOCK)$TFIRST(T);

M_RK(T)..       K(T) * (RK(T)/(IR+DELTA)) =E= K0 * F(RK(T),PL(T)) * Y(T);

M_PL(T)..       QREF(T) * PL(T)        =E= F(RK(T),PL(T)) * Y(T);

M_PKT..         SUM(TLAST, I(TLAST) - (G+DELTA)*K(TLAST))$NLPTERM
                        + SUM(TLAST, K(TLAST)*(1-DELTA) + I(TLAST) - TK)$MCPTERM =E= 0;

I_RA..          RA =E= SUM(T, PL(T)*L0*QREF(T)) + SUM(TFIRST, PK(TFIRST)*K0*KSTOCK)
                               - (TK * PKT)$MCPTERM;
```

```
TERMK$MCPTERM..        SUM(T$TLAST(T+1),  I(T+1)/I(T) - Y(T+1)/Y(T)) =E= 0;

MODEL MCP /PR_Y.Y, PR_U.U, PR_K.K, PR_I.I,
          M_P.P, M_PU.PU, M_PK.PK, M_RK.RK, M_PL.PL, M_PKT.PKT,
          I_RA.RA, TERMK.TK/;

Y.L(T)  = QREF(T);
I.L(T)  = I0 * QREF(T);
K.L(T)  = K0 * QREF(T);
P.L(T)  = PREF(T);
RK.L(T) = PREF(T) * (DELTA+IR);
PK.L(T) = PREF(T) * (1+IR);
PL.L(T) = PREF(T);
PKT.L   = SUM(TLAST, PREF(TLAST));
TK.L    = K0 * (1+G)**CARD(T);
PU.L    = PROD(T, (P.L(T)/PREF(T))**ALPHA(T) );
U.L     = ( SUM(T, PL.L(T)*L0*QREF(T)) + SUM(TFIRST, PK.L(TFIRST)*K0*KSTOCK)
         -  (TK.L * PKT.L)$MCPTERM) / PU.L;

RA.FX = U.L * PU.L;

Y.LO(T) = 1.E-5;
I.LO(T) = 1.E-5;

MCP.ITERLIM = 1000;
SOLVE MCP USING MCP;

KSTOCK = 0.80;
SOLVE MCP USING MCP;
```

$TITLE Ramsey Model: Barr-Manne Dual.

```
$INCLUDE DATA

VARIABLE
       OBJ             Objective price;

POSITIVE VARIABLES
       P(T)            Price of output,
       PC(T)           Consumption price
       PK(T)           Price of capital,
       RK(T)           Price of rental capital,
       PL(T)           Wage rate,
       PT              Price of capital in terminal period;

EQUATIONS
       OBJDEF          Objective function,
       PCDEF(T)        Defines PC
       PRF_Y(T)        Zero profit condition for output,
       PRF_I(T)        Zero profit condition for investment,
       PRF_K(T)        Zero profit condition for capital;

OBJDEF..       OBJ =E= SUM(T, ALPHA(T)*LOG(PC(T))) - SUM(T, PL(T) * QREF(T))
                              - SUM(TFIRST, PK(TFIRST)* K0 * KSTOCK);

PCDEF(T)..     PC(T) =E= P(T) / PREF(T);

PRF_Y(T)..     (RK(T)/(IR+DELTA))**KVS * (PL(T)/(1-KVS))**(1-KVS) =E= P(T);

PRF_I(T)..     P(T) =E= PK(T+1) + PT$TLAST(T);

PRF_K(T)..     PK(T) + (PT*(G+DELTA))$TLAST(T) =E= RK(T) + PK(T+1)*(1-DELTA);

MODEL DUAL /OBJDEF, PCDEF, PRF_Y, PRF_I, PRF_K /;

P.L(T)    = PREF(T);
PC.L(T)   = 1;
PK.L(T)   = PREF(T) * (1+IR);
RK.L(T)   = PREF(T) * (DELTA+IR);
PL.L(T)   = PREF(T) * (1-KVS);
```

```
PT.L      = SUM(TLAST, PREF(TLAST));


PC.LO(T)  = 0.001;
P.LO(T)   = 1E-6;
PK.LO(T)  = 1E-6;
RK.LO(T)  = 1E-6;
PL.LO(T)  = 1E-6;
PT.LO     = 1E-6;

SOLVE DUAL USING NLP MAXIMIZING OBJ;

KSTOCK = 0.80;
SOLVE DUAL USING NLP MAXIMIZING OBJ;
```

Appendix B. GAMS Code for Alternative Termination Methods.

```
$TITLE Ramsey Model: State Variable Targeting MPSGE.

$INCLUDE data

$setglobal termcnd MCP
SCALAR NLPTERM /0/, MCPTERM/0/;
%termcnd%term = 1;

$ONTEXT
$MODEL:RAMSEY

$SECTORS:
        Y(T)            ! Output
        I(T)            ! Investment
        K(T)            ! Capital stock

$COMMODITIES:
        P(T)            ! Output price
        RK(T)           ! Return to capital
        PK(T)           ! Capital price
        PL(T)           ! Wage rate
        PKT             ! Terminal capital

$CONSUMERS:
        RA              ! Representative agent

$AUXILIARY:
        TK$MCPTERM      ! Post-terminal capital stock

$PROD:Y(T) s:1
        O:P(T)           Q:1
        I:PL(T)          Q:L0
        I:RK(T)          Q:K0    P:(DELTA+IR)

$PROD:K(T)$MCPTERM
        O:PK(T+1)        Q:(1-DELTA)
        O:PKT$TLAST(T)   Q:(1-DELTA)
        O:RK(T)          Q:1
        I:PK(T)          Q:1

$PROD:K(T)$NLPTERM
        O:PK(T+1)        Q:(1-DELTA)
        O:RK(T)          Q:1
        I:PK(T)          Q:1
        I:PKT$TLAST(T)   Q:(G+DELTA)

$PROD:I(T)
        O:PK(T+1)        Q:1
        O:PKT$TLAST(T)   Q:1
        I:P(T)           Q:1

$DEMAND:RA  s:1
        D:P(T)$MCPTERM   Q:(QREF(T)*C0)  P:PREF(T)
        D:P(T)$NLPTERM   Q:C0            P:ALPHA(T)
        E:PL(T)          Q:(L0*QREF(T))
        E:PK(TFIRST)     Q:(K0*KSTOCK)
        E:PKT$MCPTERM    Q:-1         R:TK


$CONSTRAINT:TK$MCPTERM
        SUM(T$TLAST(T+1),  I(T+1)/I(T) - Y(T+1)/Y(T)) =E= 0;

$OFFTEXT
$SYSINCLUDE mpsgeset RAMSEY

Y.L(T)  = QREF(T);
I.L(T)  = I0 * QREF(T);
K.L(T)  = K0 * QREF(T);
```

```
P.L(T)  = PREF(T);
RK.L(T) = PREF(T) * (DELTA+IR);
PK.L(T) = PREF(T) * (1+IR);
PL.L(T) = PREF(T);
PKT.L   = SUM(TLAST, PREF(TLAST));
TK.L    = K0 * (1+G)**CARD(T);

RAMSEY.ITERLIM = 1000;
$INCLUDE RAMSEY.GEN
SOLVE RAMSEY USING MCP;

KSTOCK = 0.80;
$INCLUDE RAMSEY.GEN
SOLVE RAMSEY USING MCP;
```

$TITLE Ramsey Model: State Variable Targeting NLP.

```
$SYSINCLUDE gams-f
$INCLUDE DATA

*       In this code we use no adjustment of the weight on terminal consumption:

ALPHA(T) = ((1+G)/(1+IR))**(ORD(T)-1);

SCALAR  KTERM   Terminal capital stock;

*       Declare the production and utility functions here:

F(K) == (K/K0)**KVS * QREF(T)**(1-KVS);
U(C) == LOG(C);

POSITIVE VARIABLES
        K(T)            Capital stock
        C(T)            Consumption
        I(T)            Investment;

VARIABLES
        UTILITY         Utility function;

EQUATIONS
        CC(T)           Capacity constraint,
        KK(T)           Capital balance,
        KT              Terminal capital stock,
        UTIL            Discounted log of consumption: objective function;

CC(T)..         F(K(T)) =E= C(T) + I(T);

KK(T+1)..       (1-DELTA)*K(T) + I(T) =G= K(T+1);

UTIL..          UTILITY =E= -SUM(T, ALPHA(T)* U(C(T)));

KT..            KTERM =E= SUM(TLAST, I(TLAST) + (1-DELTA) * K(TLAST));

MODEL PRIMAL /CC, KK, KT, UTIL/;

C.L(T)       = C0;
K.LO(T)      = 1E-4;
C.LO(T)      = 1E-4;
I.LO(T)      = 1E-4;
K.FX(TFIRST) = K0;

KTERM = K0 * (1 + G)**CARD(T);

SOLVE PRIMAL MINIMIZING UTILITY USING NLP;

*       Experiment:

K.FX(TFIRST) = K0*0.80;

*       Initial guess on terminal capital:
```

```
KTERM = K0 * (1 + G)**CARD(T);

*       Evaluate the side constraint:

PARAMETER       ITERLOG         ITERATION LOG
                DFDK            Evaluation of gradient;

SET     ITER /IT0*IT6/;

LOOP(ITER,

        ITERLOG(ITER,"KTERM") = KTERM;

        SOLVE PRIMAL MINIMIZING UTILITY USING NLP;
        LOOP(TLAST(T), ITERLOG(ITER,"F") = I.L(T)/I.L(T-1) - F(K.L(T))/F(K.L(T-1)));

        KTERM = KTERM * 1.05;
        SOLVE PRIMAL MINIMIZING UTILITY USING NLP;

        LOOP(TLAST(T), ITERLOG(ITER,"F'") = I.L(T)/I.L(T-1) - F(K.L(T))/F(K.L(T-1)));

*       Take the standard Newton iterate:

        ITERLOG(ITER,"DFDK") = (ITERLOG(ITER,"F'")-ITERLOG(ITER,"F"))/(0.05*KTERM/1.05);

        KTERM = KTERM/1.05 - ITERLOG(ITER,"F")/ITERLOG(ITER,"DFDK");
);
```

Appendix C. GAMS Code for Model with Multiple Agents.

```
$TITLE Ramsey Model with Multiple Regions, MPSGE.

$INCLUDE DATA

SET     R       /N,S/;

ALIAS (R,RR);

PARAMETER
        X0(R)   Base year export levels across regions,
        M0(R)   Base year import levels across regions,
        KS(R)   Capital stock index across regions;

X0(R)  = 0.1;
M0(R)  = X0(R++1);
KS(R)  = 1;

$ONTEXT
$MODEL:RAMSEY

$SECTORS:
        Y(R,T)          ! Output across regions
        I(R,T)          ! Investment across regions
        K(R,T)          ! Capital stock across regions
        X(R,T)          ! Export index across regions
        C(R,T)          ! Private consumption across regions
        U(R)            ! Intertemporal welfare index across regions

$COMMODITIES:
        P(R,T)          ! Output price across regions
        RK(R,T)         ! Return to capital across regions
        PK(R,T)         ! Capital price across regions
        PL(R,T)         ! Wage rate across regions
        PM(R,T)         ! Import price across regions
        PC(R,T)         ! Price of private consumption across regions
        PU(R)           ! Intertemporal welfare price index across regions
        PKT(R)          ! Terminal capital across regions

$CONSUMERS:
        RA(R)           ! Representative agent across regions

$AUXILIARY:
        TK(R)           ! Post-terminal capital stock across regions
        TA(R)           ! Terminal adjustment for changes in assets

$PROD:Y(R,T) s:1
        O:P(R,T)        Q:(1+M0(R))
        I:PL(R,T)       Q:L0
        I:RK(R,T)       Q:K0            P:(DELTA+IR)
        I:PM(R,T)       Q:M0(R)

$PROD:X(R,T)
        O:PM(RR,T)$(ORD(RR) NE ORD(R))          Q:1
        I:P(R,T)        Q:1

$PROD:K(R,T)
        O:PK(R,T+1)             Q:(1-DELTA)
        O:PKT(R)$TLAST(T)       Q:(1-DELTA)
        O:RK(R,T)               Q:1
        I:PK(R,T)               Q:1

$PROD:I(R,T)
        O:PK(R,T+1)             Q:1
        O:PKT(R)$TLAST(T)       Q:1
        I:P(R,T)                Q:1

$PROD:C(R,T)
        O:PC(R,T)       Q:1
```

```
       I:P(R,T)       Q:1

$PROD:U(R) s:1
       O:PU(R)        Q:(SUM(T, PREF(T)*QREF(T)*C0))
       I:PC(R,T)      Q:(QREF(T)*C0) P:PREF(T)

$DEMAND:RA(R)
       D:PU(R)        Q:(SUM(T, PREF(T)*QREF(T)*C0))
       E:PL(R,T)      Q:(L0*QREF(T))
       E:PK(R,TFIRST) Q:(K0*KS(R))
       E:PKT(R)       Q:-1           R:TK(R)
       E:PU(R)        Q:-1           R:TA(R)

$CONSTRAINT:TK(R)
        SUM(T$TLAST(T+1),  I(R,T+1)/I(R,T) - Y(R,T+1)/Y(R,T)) =E= 0;

$CONSTRAINT:TA(R)
       TA(R) =E= (SUM(T$TLAST(T), P(R,T)*C(R,T) - PL(R,T)*QREF(T)*L0) /
                      SUM(T$TLAST(T), SUM(RR, P(RR,T)*C(RR,T) - PL(RR,T)*L0*QREF(T) ))) *
                      SUM(RR, PKT(RR)*TK(RR)) - PKT(R)*TK(R);

$OFFTEXT
$SYSINCLUDE mpsgeset RAMSEY

U.L(R)    = 1;
PU.L(R)   = 1;
Y.L(R,T)  = QREF(T);
I.L(R,T)  = I0 * QREF(T);
K.L(R,T)  = K0 * QREF(T);
C.L(R,T)  = C0 * QREF(T);
X.L(R,T)  = X0(R) * QREF(T);
TK.L(R)   = K0 * (1+G)**CARD(T);
TA.L(R)   = 0;
P.L(R,T)  = PREF(T);
RK.L(R,T) = PREF(T) * (DELTA+IR);
PK.L(R,T) = PREF(T) * (1+IR);
PL.L(R,T) = PREF(T);
PC.L(R,T) = PREF(T);
PM.L(R,T) = PREF(T);
PKT.L(R)  = SUM(TLAST, PREF(TLAST));

TA.LO(R)  = -INF;

RAMSEY.ITERLIM = 0;
$INCLUDE RAMSEY.GEN
SOLVE RAMSEY USING MCP;

RAMSEY.ITERLIM = 1000;
KS("N") = 0.80;
$INCLUDE RAMSEY.GEN
SOLVE RAMSEY USING MCP;
```

$TITLE Ramsey Model with Multiple Regions, MCP.

```
$SYSINCLUDE GAMS-F
$INCLUDE DATA

ALPHA(T) = ((1+G)/(1+IR))**(ORD(T)-1);
ALIAS (T,TT);
ALPHA(T) = ALPHA(T) / SUM(TT, ALPHA(TT));

SET     R       /N,S/;
ALIAS (R,RR);

PARAMETER
       X0(R)  Base year export levels across regions,
       M0(R)  Base year import levels across regions,
       KS(R)  Capital stock index across regions;

X0(R)  = 0.1;
```

```
M0(R)  = X0(R++1);
KS(R)  = 1;

PARAMETER
       CVS(R)  Capital value share across regions,
       LVS(R)  Labor value share across regions,
       IVS(R)  Import value share across regions;

CVS(R) = KVS / (1+M0(R));
LVS(R) = (1-KVS) / (1+M0(R));
IVS(R) = M0(R) / (1+M0(R));

*       Declare the production and utility functions here:

F(RK(R),PL(R),PM(R)) == (RK(R)/(IR+DELTA))**CVS(R) * PL(R)**LVS(R) * PM(R)**IVS(R);
UT(P(R)) == PROD(T, (P(R,T)/PREF(T))**ALPHA(T));

POSITIVE VARIABLES
       Y(R,T)          Output across regions,
       I(R,T)          Investment across regions,
       K(R,T)          Capital stock across regions,
       X(R,T)          Export index across regions,
       C(R,T)          Private consumption across regions,
       U(R)            Intertemporal welfare index across regions,

       P(R,T)          Output price across regions,
       RK(R,T)         Return to capital across regions,
       PK(R,T)         Capital price across regions,
       PL(R,T)         Wage rate across regions,
       PM(R,T)         Import price across regions,
       PC(R,T)         Price of private consumption across regions,
       PU(R)           Intertemporal welfare price index across regions,
       PKT(R)          Terminal capital across regions,

       RA(R)           Representative agent across regions,
       TK(R)           Post-terminal capital stock across regions,
       TA(R)           Terminal adjustment for changes in assets;

EQUATIONS
       PR_Y(R,T)       Zero profit condition for output,
       PR_I(R,T)       Zero profit condition for investment,
       PR_K(R,T)       Zero profit condition for capital,
       PR_X(R,T)       Zero profit condition for export,
       PR_C(R,T)       Zero profit condition for private consumption,
       PR_U(R)         Zero profit condition for intertemporal welfare,

       M_P(R,T)        Market clearing for output,
       M_RK(R,T)       Market clearing for rental capital,
       M_PK(R,T)       Market clearing for capital,
       M_PL(R,T)       Market clearing for labor,
       M_PM(R,T)       Market clearing for export,
       M_PC(R,T)       Market clearing for private consumption,
       M_PU(R)         Market clearing for intertemporal welfare,
       M_PKT(R)        Market clearing for terminal capital,

       I_RA(R)         Income balance for representative agents,

       TERMK(R)        Terminal constraint for capital stock,
       TERMA(R)        Terminal constraint for assets;

PR_Y(R,T)..    F(RK(R,T),PL(R,T),PM(R,T)) =E= P(R,T);

PR_U(R)..      UT(P(R)) =E= PU(R);

PR_K(R,T)..    PK(R,T) =E= RK(R,T) + (1-DELTA)*(PK(R,T+1) + PKT(R)$TLAST(T));

PR_I(R,T)..    P(R,T) =E= PK(R,T+1) + PKT(R)$TLAST(T);

PR_C(R,T)..    P(R,T) =E= PC(R,T);

PR_X(R,T)..    P(R,T) =E= PM(R++1,T);
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M_P(R,T)..      Y(R,T)*(1+M0(R)) =E= C(R,T) + I(R,T) + X(R,T);

M_PU(R)..       U(R) * SUM(T, PREF(T)*QREF(T)*C0) =E= RA(R) / PU(R);

M_PK(R,T)..     K(R,T) =E= (1-DELTA)*K(R,T-1) + I(R,T-1) + (K0*KS(R))$TFIRST(T);

M_RK(R,T)..     K(R,T) * (RK(R,T)/(IR+DELTA)) =E= K0 * F(RK(R,T),PL(R,T),PM(R,T)) * Y(R,T);

M_PL(R,T)..     QREF(T) * PL(R,T) =E= F(RK(R,T),PL(R,T),PM(R,T)) * Y(R,T);

M_PC(R,T)..     C(R,T) =E= ALPHA(T) * PU(R) * U(R) * SUM(TT, PREF(TT)*QREF(TT)*C0) / PC(R,T);

M_PM(R,T)..     X(R++1,T) * PM(R,T) =E= X0(R) * F(RK(R,T),PL(R,T),PM(R,T)) * Y(R,T);

M_PKT(R)..      SUM(TLAST, K(R,TLAST)*(1-DELTA) + I(R,TLAST) - TK(R)) =E= 0;

I_RA(R)..       RA(R) =E= SUM(T, PL(R,T)*L0*QREF(T)) + SUM(TFIRST, PK(R,TFIRST)*K0*KS(R))
                               - TK(R) * PKT(R);

TERMK(R)..      SUM(T$TLAST(T+1),  I(R,T+1)/I(R,T) - Y(R,T+1)/Y(R,T)) =E= 0;

TERMA(R)..      TA(R) =E= (SUM(T$TLAST(T), P(R,T)*C(R,T) - PL(R,T)*QREF(T)*L0) /
                       SUM(T$TLAST(T), SUM(RR, P(RR,T)*C(RR,T) - PL(RR,T)*L0*QREF(T) ))) *
                       SUM(RR, PKT(RR)*TK(RR)) - PKT(R)*TK(R);

MODEL MCP /PR_Y.Y, PR_U.U, PR_K.K, PR_I.I, PR_C.C, PR_X.X,
          M_P.P, M_PU.PU, M_PK.PK, M_RK.RK, M_PL.PL, M_PC.PC, M_PM.PM, M_PKT.PKT,
          I_RA.RA, TERMK.TK, TERMA.TA/;

Y.L(R,T)  = QREF(T);
I.L(R,T)  = I0 * QREF(T);
K.L(R,T)  = K0 * QREF(T);
C.L(R,T)  = C0 * QREF(T);
X.L(R,T)  = M0(R) * QREF(T);
P.L(R,T)  = PREF(T);
RK.L(R,T) = PREF(T) * (DELTA+IR);
PK.L(R,T) = PREF(T) * (1+IR);
PL.L(R,T) = PREF(T);
PC.L(R,T) = PREF(T);
PM.L(R,T) = PREF(T);
PKT.L(R)  = SUM(TLAST, PREF(TLAST));
PU.L(R)   = 1;
U.L(R)    = 1;
TK.L(R)   = K0 * (1+G)**CARD(T);
TA.L(R)   = 0;
RA.L("S") = SUM(T, PL.L("S",T)*L0*QREF(T)) + SUM(TFIRST, PK.L("S",TFIRST)*K0*KS("S"))
               -  TK.L("S") * PKT.L("S");
RA.FX("N")= SUM(T, PL.L("N",T)*L0*QREF(T)) + SUM(TFIRST, PK.L("N",TFIRST)*K0*KS("N"))
               -  TK.L("N") * PKT.L("N");

Y.LO(R,T) = 1.E-5;
I.LO(R,T) = 1.E-5;
TA.LO(R)  = -INF;

MCP.ITERLIM = 0;
SOLVE MCP USING MCP;

MCP.ITERLIM = 1000;
KS("N") = 0.80;
SOLVE MCP USING MCP;
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