

Dynamic General Equilibrium with GAMS/MPSGE

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Lecture Notes Prepared for the
UNSW Workshop, February 24-27, 2004

This document is based on a one-day short-course lecture on dynamic general equilibrium analysis with MPSGE. I begin by going through the logic of the Ramsey model which is often presented as a dynamic optimization problem:

$$\max \sum_{t=0}^{\infty} \left(\frac{1}{1+\rho} \right)^t \frac{C_t^{1-\theta} - 1}{1-\theta}$$

s.t.

$$\begin{aligned} C_t &= f(K_t) - I_t \\ K_{t+1} &= (1-\delta)K_t + I_t \\ K_0 &= \bar{K}_0 \end{aligned}$$

The maximand in this problem is often called constant-elasticity-of-intertemporal substitution (CEIS) utility function. As will be shown below, it simply represents a monotonic transformation of conventional CES utility function.

Here, as in many macroeconomics textbooks, aggregate output is expressed as a function of the capital stock alone, i.e.:

$$Y_t = f(K_t)$$

In the MPSGE representation of the Ramsey model, it is convenient to work with a constant-returns production function in which we have inputs of both labor and capital:

$$Y_t = F(\bar{L}_t, K_t)$$

When labor is in fixed supply, the production function exhibits diminishing returns to capital. There is therefore no loss of generality by formulating the model on the basis of a constant returns to scale technology.

In writing down a model it is helpful to employ the *unit cost function* associated with the production function $F()$:¹

$$c(p_t^L, r_t^K) \equiv \min p_t^L a_L + r_t^K a_K$$

s.t.

$$F(a_L, a_K) = 1$$

¹Note that the lower-case function $c()$ represents unit cost, while the upper case C_t represents consumption in year t . In the equilibrium model $C_t(p, M)$ represents the demand for output in year t as a function of output prices and aggregate present value of income.

Shephard's lemma tells us that the *compensated demand functions* for labor and capital are the partial derivatives of the unit cost function:

$$a_K(r^K, p^L) = \frac{\partial c(p_t^L, r_t^K)}{\partial r_t^K}$$

and

$$a_L(r^K, p^L) = \frac{\partial c(p_t^L, r_t^K)}{\partial p_t^L}$$

The representative agent model can be formulated as a general equilibrium model which is completely routine, apart from the fact that there are an infinite-number of variables. Following the conventional GAMS/MPSGE framework, equilibrium in the model is characterized by three classes of equations:

1. Market clearance conditions and associated market prices are as follows:²

- Output market (market price p_t):

$$Y_t = C_t(p, M) + I_t$$

- Labor market (wage rate p_t^L):

$$\bar{L}_t = a_L(r_t^K, p_t^L) Y_t$$

- Market for capital services (capital rental rate r_t^K):

$$K_t = a_K(r_t^K, p_t^L) Y_t$$

- Capital stock (capital purchase price p_t^K):

$$K_{t+1} = (1 - \delta)K_t + I_t$$

2. Zero profit conditions and associated activities are:³

- Output (Y_t):

$$p_t = c(p_t^L, r_t^K)$$

- Investment ($I_t \geq 0$):

$$p_t \geq p_{t+1}^K$$

- Capital stock (K_t):

$$p_t^K = r_t^K + (1 - \delta)p_{t+1}^K$$

3. Income balance:

$$M = p_0^K \bar{K}_0 + \sum_{t=0}^{\infty} p_t^L \bar{L}_t$$

Two questions might arise for an MPSGE modeller looking at this equilibrium model. First, the careful observer might note that the demand functions, $C_t(p, M)$ have not been specified, and because these arise from CES preferences so there may be some details to work out. This problem is considerably easier than the second issue, namely *how do we solve an infinite-dimensional system of nonlinear equations*. Let's first consider the issue of CEIS preferences.

Consider the following *equivalent* representations of intertemporal preferences:

²The demand functions employed in this model assure that all prices will be nonzero in equilibrium. There is no formal need, therefore, to associated prices with market clearance conditions, as would be required in a conventional complementarity problem. I provide an associated here in order to help understand how the model might be extended with demand functions which would admit zero prices.

³The only activity level which could possibly fall to zero would be investment, and that would only happen in a policy scenario which resulted in a substantial reduction in the return to capital.

1. Additively separable utility:

$$U(c) = \sum_{t=0}^{\infty} \left(\frac{1}{1+\rho} \right)^t \frac{C_t^{1-\theta} - 1}{1-\theta}$$

2. Linearly homogeneous utility:

$$\hat{U}(c) = \left[\sum_{t=0}^{\infty} \left(\frac{1}{1+\rho} \right)^t C_t^{1-\theta} \right]^{1/1-\theta}$$

It is possible to determine the equivalence of U and \hat{U} by recalling that a monotonic transformation of utility does not alter the underlying preference ordering. Observe that:

$$\hat{U} = V(U) = [aU + \kappa]^{1/a}$$

where

$$\kappa = \sum_{t=0}^{\infty} \left(\frac{1}{1+\rho} \right)^t = \frac{1+\rho}{\rho},$$

and

$$a = 1 - \theta.$$

$V()$ is a monotonic transformation ($V' > 0$), hence optimization of U and \hat{U} yield identical demand functions.

Alternatively, recall that preference orderings are defined by the *marginal rate of substitution*. In both of these models we have:

$$\frac{\partial U / \partial C_{t+1}}{\partial U / \partial C_t} = \frac{1}{1+\rho} \left(\frac{C_t}{C_{t+1}} \right)^{1-\theta}$$

There are several advantages associated with the use of linearly homogeneous representation. First of all, these preferences can be represented in MSPGE. Second, the reporting of welfare changes as Hicksian-equivalent variations is trivial with \hat{U} – a 1% change in \hat{U} corresponds to a 1% equivalent variation in income.

CEIS preferences *over a finite horizon* can be represented in MPSGE as follows:

```
$prod:C s:sigma
o:PC q:(c0*sum(t,pref(t)*qref(t)))
i:P(t) q:(c0*qref(t)) p:pref(t)
```

Intertemporal preferences in an MPSGE model are typically based on the following parameters:

- **sigma** is the intertemporal elasticity of substitution,
- **c0** is the base year consumption level, and
- **qref(t) = (1+g0)**(ord(t)-1)** is the *baseline equilibrium index of economic activity*, calculated on the basis of a steady-state growth rate equal to **g0**.
- **pref(t) = (1+r0)**(ord(t)-1)** is the *baseline present value price path*, calculated on the basis of a steady-state interest rate equal to **r0**.

The MPSGE representation includes a discount rate which is define *implicitly* as:

$$\rho = \frac{1 + \bar{r}}{(1 + g)^{1-\theta}} - 1$$

In order to solve a finite approximation of the model with a T -period model horizon, we need to *decompose* the consumer's problem. Consider the infinite-horizon problem of the representative agent in Ramsey's model:

$$\max \sum_{t=0}^{\infty} \left(\frac{1}{1 + \rho} \right)^t u(c_t)$$

s.t.

$$\sum_{t=0}^{\infty} p_c c_t = p_0^K \bar{K}_0 + \sum_{t=0}^{\infty} p_t^L \bar{L}_t$$

in which $u(c) = \frac{c^{1-\theta}-1}{1-\theta}$, and define a value of terminal assets to be:

$$A_T^* = \sum_{t=T+1}^{\infty} (p_c c_t^* - p_t^L \bar{L}_t)$$

Then consider the equivalent model:

$$\max \sum_{t=0}^T \left(\frac{1}{1 + \rho} \right)^t u(c_t) + \sum_{t=T+1}^{\infty} \left(\frac{1}{1 + \rho} \right)^t u(c_t)$$

s.t.

$$\sum_{t=0}^T p_c c_t = p_0^K \bar{K}_0 + \sum_{t=0}^T p_t^L \bar{L}_t - A_T$$

$$\sum_{t=T+1}^{\infty} p_c c_t = A_T + \sum_{t=T+1}^{\infty} p_t^L \bar{L}_t$$

If A_T is fixed then this can be posed as two separate optimization problems, one running through time period T and another for the post-terminal period. When terminal assets are assigned a value of A_T^* , corresponding to the infinite-horizon solution, then the finite horizon model will then produce consumption levels for years 0 through T which are identical to the ∞ -horizon model.

How do we find A_T^ ?*

Terminal assets in the closed economy model are simply equal to the value of the capital stock at the start of period $T + 1$. The model running through year T then produces a good approximation to the consumer problem when we have a good approximation to the terminal capital stock. The key insight provided by Lau, Pahlke and Rutherford (2000) is that the state variable K_{T+1} can be determined as part of the equilibrium calculation by *targeting* the associated control variable, I_T . In the present model this could be based on any of the following *primal constraints*:

- Terminal investment growth rate set equal to the long-run steady-state growth rate:

$$I_T / I_{T-1} = 1 + g$$

- Terminal investment growth rate set equal to the growth rate of aggregate output:

$$I_T/I_{T-1} = Y_T/Y_{T-1}$$

- Terminal investment growth rate set equal to the growth rate of consumption:

$$I_T/I_{T-1} = C_T/C_{T-1}$$

State-variable targeting provides a very compact means of determining the terminal capital stock. In models with multiple consumers living beyond period T , it would be necessary to account for which of these agents owns the assets. Note that some agents may have *negative* asset positions at the end of the model – particularly in overlapping generations models where young households accumulate debt which is repaid in middle age.

The final detail involved in implementing a dynamic model in MPSGE is *calibration*. The simplest approach to set up the model along a steady-state growth rate in which the interest rate (\bar{r}) and growth rate (\bar{g}) are given. The first thing to work out is to determine the structure of the benchmark equilibrium.

Here are the steps involved in sorting out the steady-state conditions which related investment and capital earnings in a static dataset which is consistent with a steady-state growth path:

1. The zero-profit condition for I_t reveals the price level for capital:

$$p_{t+1}^K = \frac{p_t^K}{1 + \bar{r}} = p_t$$

hence

$$p_t^K = (1 + \bar{r})p_t$$

The base year price of capital is then:

$$\bar{p}^K = 1 + \bar{r}$$

2. The zero profit condition for K_t determines the price level for r_t^K :

$$p_t^K = r_t^K + (1 - \delta)p_{t+1}^K$$

Substituting the values of p_t^K and p_{t+1}^K reveals that the base year rental price of capital is sufficient to cover interest plus depreciation:

$$\bar{r}^K = \bar{r} + \delta$$

3. The main challenge involved in calibrating a dynamic model centers on the reconciliation of base year capital earnings, investment, the steady-state interest rate and the capital depreciation rate. To see how this works, consider the market clearance condition for capital in the first period:

$$K_1 = \bar{K}_0(1 - \delta) + \bar{I} = (1 + g)\bar{K}_0$$

This implies that base year investment can be calculated on the basis of growth and depreciation of the base year capital stock:

$$\bar{I} = \bar{K}_0(g + \delta)$$

Finally, we can use \bar{r}^K to determine \bar{K}_0 on the basis of the value of capital earnings in the base year, $\bar{V}K$, hence:

$$\bar{I} = \bar{V}K \frac{g + \delta}{r + \delta}$$

The problem that arises in applied models is that \bar{I} and $\bar{V}K$ will not satisfy this relation for arbitrary values of \bar{g} , \bar{r} and δ . Something typically has to be adjusted to match up the dataset with the baseline growth path.

The GAMS code on the following page presents a GAMS/MPSGE model which has been formulated following these ideas.

1 GAMS Code Describing a Steady-State Equilibrium

This code reads data describing a base year equilibrium. Investment levels are imputed from the base year capital stock which is in-turn inferred from the assumed capital value share.

```

1 $TITLE Ramsey Model -- MPSGE formulation calibrated to base year data
2
3 $ontext
4 Calibrate to the steady-state condition:
5
6 IO = KDO * (g + delta) / (r + delta)
7
8 where g=2, delta=7, r=5, so
9
10 IO = 48 * 9 / 12 = 36
11
12      Y      I      FD
13 P      100    -36    -64
14 PL     -52
15 RK     -48      48
16 PS           36    -36
17
18 $offtext
19
20 PARAMETER      g      Growth rate      /0.02/,
21                r      Interest rate     /0.05/,
22                delta  Depreciation rate  /0.07/,
23                kvs    Capital value share /0.48/,
24                y0     Base year output,
25                kd0    Base year rental value of capital,
26                k0     Base year capital stock,
27                i0     Base year investment,
28                c0     Base year consumption,
29                l0     Base year labor input,
30                kstock Base year capital stock multiplier /1/,
31                taxk(t) Capital tax rate in period T,
32                qref(t) Reference quantity path,
33                pref(tt) Reference price path;
34
35 *      Use the GAMS ORD (ordinality) and CARD (cardinality)
36 *      functions to automate the identification of the first
37 *      and last periods of the model horizon:
38
39 t0(t)      = yes$(ord(t) eq 1);
40 t1(t)      = yes$(ord(t) eq card(t));
41 tterm(tt)  = yes$(ord(tt) eq card(tt));
42
43 *      Calibrate the model to the baseline growth path:
44
45 y0 = 100;
46 kd0 = kvs * y0;
47 l0 = y0 - kd0;
48 k0 = kd0 / (r + delta);
49 i0 = (g + delta) * k0;
50 c0 = y0 - i0;
51 taxk(t) = 0;

```

```

52 qref(t) = (1+g)**(ord(t)-1);
53 pref(tt) = (1/(1+r))**(ord(tt)-1);
54
55 DISPLAY y0, kd0, l0, k0, i0, c0, g, r, delta;

```

2 GAMS Code for Benchmark Replication of the Model

This section of the code declares the GAMS/MPSGE model, assigns steady-state values for activity levels and price, and then checks consistency of the resulting model.

```

1
2 *      Define the time horizon of the model using two sets, one which
3 *      includes the post-terminal year and one which covers only the
4 *      endogenous years:
5
6 SET    tt      Time horizon (with the first year of the post-terminal period) /2004*2081/,
7        t(tt)   Time period over the model horizon /2004*2080/;
8        t0(t)   Year 0,
9        t1(t)   Last endogenous year,
10       tp(tt)  First post-terminal year;
11
12 $ONTEXT
13
14 $MODEL:RAMSEY
15
16 $SECTORS:
17     Y(t)      !      Output
18     I(t)      !      Investment
19     K(t)      !      Capital stock
20
21 $COMMODITIES:
22     P(t)      !      Output price
23     RK(t)     !      Return to capital
24     PK(tt)    !      Capital price
25     PL(t)     !      Wage rate
26
27 $CONSUMERS:
28     RA        !      Representative agent
29
30 $AUXILIARY:
31     TK        !      Post-terminal capital stock
32
33 $PROD:Y(t) s:1
34     O:P(t)    Q:Y0
35     I:PL(t)   Q:L0
36     I:RK(t)   Q:KDO      A:RA      T:TaxK(t)
37
38 $PROD:K(tt)$T(tt)
39     O:PK(TT+1) Q:(KO*(1-DELTA))
40     O:RK(tt)   Q:KDO
41     I:PK(tt)   Q:KO
42
43 $PROD:I(tt)$T(tt)
44     O:PK(TT+1) Q:I0
45     I:P(tt)    Q:I0
46
47 $DEMAND:RA s:1
48     D:P(t)    Q:(qref(t)*C0) P:pref(t)
49

```

```

50      E:PL(t)          Q:(L0*qref(t))
51      E:PK(T0)       Q:(K0*KSTOCK)
52      E:PK(TP)       Q:-1    R:TK
53
54 $REPORT:
55      V:C(t)  D:P(t)  DEMAND:RA
56      V:W    W:RA
57
58 $CONSTRAINT:TK
59      SUM(T$TL(T+1), I(T+1)/I(t) - Y(T+1)/Y(t)) =E= 0;
60
61 $OFFTEXT
62 $SYSINCLUDE mpsgeset RAMSEY
63
64 *      Assign steady-state equilibrium values for quantities
65 *      and prices:
66
67 Y.L(t) = qref(t);
68 I.L(t) = qref(t);
69 K.L(t) = qref(t);
70
71 P.L(t) = pref(t);
72 RK.L(t) = pref(t);
73 PL.L(t) = pref(t);
74
75 *      The steady-state price of capital is the output price
76 *      times one plus the interest rate:
77
78 PK.L(tt) = (1+r) * pref(tt);
79 TK.L      = k0 * (1+g)**card(t);
80
81 RAMSEY.ITERLIM = 0;
82 $INCLUDE RAMSEY.GEN
83 SOLVE RAMSEY USING MCP;
84 ABORT$(RAMSEY.OBJVAL > 0.001) "Model does not calibrate.";
85 RAMSEY.ITERLIM = 10000;

```

3 GAMS Code for a Policy Experiment

This code assigns a tax on capital earnings beginning in year 6. The resulting equilibrium is computed assuming that economic agents anticipate the application of the tax, resulting in a sharp response in investment and other economic variables to the new economic environment. Over time the tax leads to a reduction in the steady-state capital stock and the real wage.

Output are presented in graphs using GNUPLOT, both to the windows screen and as encapsulated postscript.

```

1
2 *      Apply a tax on capital inputs of 25% beginning in year 6:
3
4 TAXK(t)$ (ORD(t) > 5) = 0.25;
5
6 $INCLUDE RAMSEY.GEN
7 SOLVE RAMSEY USING MCP;
8
9 *      Generate some reports with graphs:
10
11 PARAMETER      indices      "Consumption, Investment and Capital Stock Indices";
12 indices(t,"C") = C.L(t)/(c0*qref(t));

```



```

13 indices(t,"I") = I.L(t)/qref(t);
14 indices(t,"K") = K.L(t)/qref(t);
15
16 DISPLAY INDICES;
17
18 *      Define the domain over which the X-axis will be defined:
19
20 $setglobal domain t
21
22 *      Define the labels to be printed along the X axis:
23
24 set      tlbl(t) Time periods to be labelled in output plots /2010,2030,2050,2070/;
25
26 $setglobal labels tlbl
27
28
29 *      Place the key to the figures outside the graph (see GNUPLOT 3.7 Help File):
30
31 $setglobal gp_opt0 "set key outside"
32
33 *      Plot the graph with horizontal and vertical grid lines (see GNUPLOT 3.7 Help File):
34
35 $setglobal gp_opt1 "set grid"
36
37 *      Generate the plot to the screen -- it can subsequently be copied to the clipboard
38 *      using a right-click of the mouse, and then pasted into a separate program for
39 *      publication:
40
41 $libinclude plot indices
42
43 *      Repeat the report generation process a couple more times:
44
45 parameter      price      "Capital prices and wage rate";
46 price(t,"RK") = RK.L(t)/P.L(t);
47 price(t,"PK") = PK.L(t)/((1+r)*P.L(t));
48 price(t,"PL") = PL.L(t)/P.L(t);
49
50
51 $libinclude plot price
52
53 parameter grates(t,*) Growth rates through the transition;
54
55 grates(t,"c") = 100 * (C.l(t+1)/C.l(t) - 1);
56 grates(t,"i") = 100 * (I.l(t+1)/I.l(t) - 1);
57 grates(t,"k") = 100 * (K.l(t+1)/K.l(t) - 1);
58
59 set gs /c,i,k/;
60 grates(tl,gs) = na;
61 $libinclude plot grates
62
63 *      Produce some EPS graphs to put into the notes:
64
65 $setglobal batch yes
66 $setglobal gp_opt2 "set term postscript eps monochrome 'Times-Roman' 16"
67 $setglobal gp_opt3 "set title"
68 $setglobal gp_opt0 "set key outside width 4"
69
70 $setglobal gp_opt4 "set output 'price.eps'"
71 $libinclude plot price
72 $setglobal gp_opt4 "set output 'indices.eps'"
73 $libinclude plot indices
74 $setglobal gp_opt4 "set output 'grates.eps'"
75 $libinclude plot grates

```

Figure 1: Consumption, Investment and Capital Stock Indices

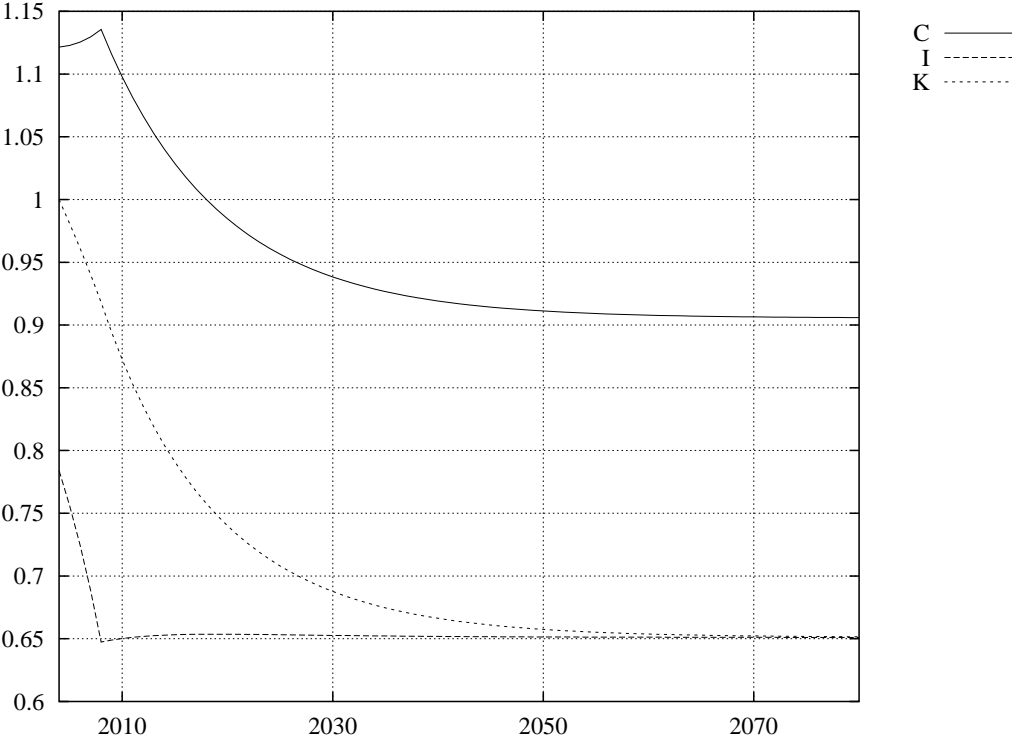


Figure 2: Capital Prices and Wage Rates

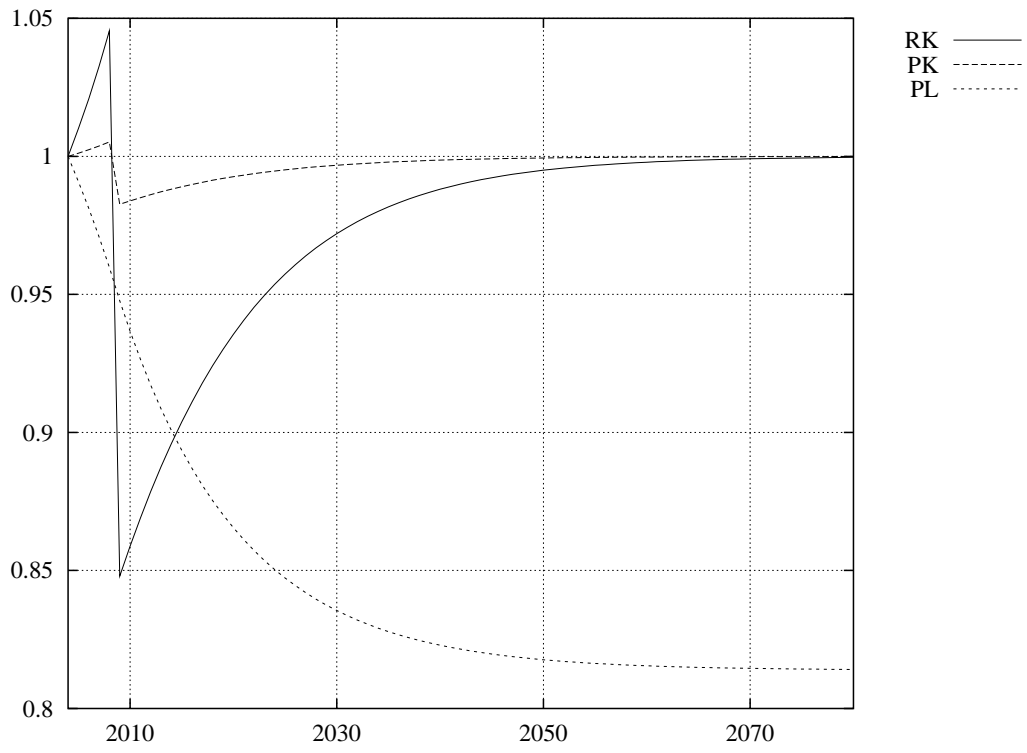


Figure 3: Growth Rates through the Transition

