Calibrated CES Utility Functions: A Worked Example

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March 24, 2008

> Thomas lives in Ann Arbor where he currently spends 30% of his income on rent. He has an employment offer in Zürich which pays 50% more than he currently earns, but he is hesitant to take the job because rental rates in Zürich are three times higher than in Ann Arbor. Assume that Thomas has CES preferences with elasticity of substitution σ. On purely economic grounds, should he move?

As is the case for all interesting questions in economics, the only good answer to this problem is "It depends.". It is fairly easy to see how this works on intuitive grounds. Thomas's offer in Zürich does not pay him enough to live exactly the lifestyle that he enjoys in Ann Arbor, as he would need a 60% raise to cover rent and consumption. The elasticity of substitution is key. If it is high, he more willing substitutes consumption of goods and services for housing and thereby lowers his cost of living in Zürich. On the other hand, if the elasticity is low, he is "stuck in his ways", and the move is a bad idea.

Before thinking about how to solve this model I want to reflect on how we proceed. We are given information about Thomas's choices in Ann Arbor. This information is essentially an observation of a *benchmark equilibrium*, consisting of the prevailing prices and quantities of goods demand. The benchmark equilibrium data together with assumptions about elasticities are used to evaluate Thomas's choices after a discrete change in the economic environment. The steps involved in solving this little textbook model are identical to those typically employed in applied general equilibrium analysis. A general equilibrium model involves the simultaneous interaction of agents on both sides of each market, a benchmark equilibrium dataset describes all relevant economic transactions in a reference period, and calibrated functions are used to characterize choices by consumers and producers following a discrete change in the economic environment. Economic impacts are evaluated on the basis of observed choices and assumed elasticities.

*I wish to thank Torben Mideksa and Todd Caldis for the comments and corrections they communicated based on on an earlier draft of this paper. I remain responsible for remaining errors and inconsistencies. Prices and income levels employed in this exposition were arbitrarily defined. Any resemblence to the actual circumstances of specific individuals is purely coincidental.

1 Graphical Analysis

The idea that Thomas's decision to move depends on his willingness to substitute goods for housing is illustrated in Figure 1. In this figure, Thomas's choices in Ann Arbor are depicted as point a. This point on the budget constraint is shown with three alternative benchmark indifference curves, labelled as σ_H (high elasticity), σ_L (low elasticity) and σ^* (critical elasticity).

The budget constraint which Thomas faces in Zürich is portrayed as the steeper line which lies to the left of point a. Thomas's consumption basket in Ann Arbor is unaffordable in Zürich, so the consumption point a is outside the Zürich budget set.

If Thomas were to move, where would he choose to consume on the new budget line? This depends on the nature of his preferences. If he has a high elasticity of substitution, he could drastically reduce his housing and increase his demand for other goods. If he has low elasticity of substitution, he may choose to lower his consumption of both goods. (In the extreme case of Leontief preferences, his consumption point would be at the intersection of the new budget and a line from a to the origin.)

The "critical elasticity", σ^*, represents how Thomas's preferences would be configured if he were completely indifferent to the move. He would then live in Zürich with lower housing demand and increased consumption of other goods. The indifference curve tangent at b is the same which is tangent at a.

2 Demand Functions

The algebraic solution to this model begins by adopting a formal representation of preferences. We can write Thomas's utility function as:

$$U(C,H) = (\alpha C^\rho + (1-\alpha)H^\rho)^{1/\rho} \tag{1}$$

in which ρ is defined by the elasticity of substitution, σ, as

$$\rho = 1 - 1/\sigma.$$

The formal model of consumer choice involves solution of the budget-constrained utility maximization problem:

$$\max U(C,H) \text{ s.t. } C + p_H H = M \tag{2}$$

(Note: we normalize prices so that the price of other goods is unity.)

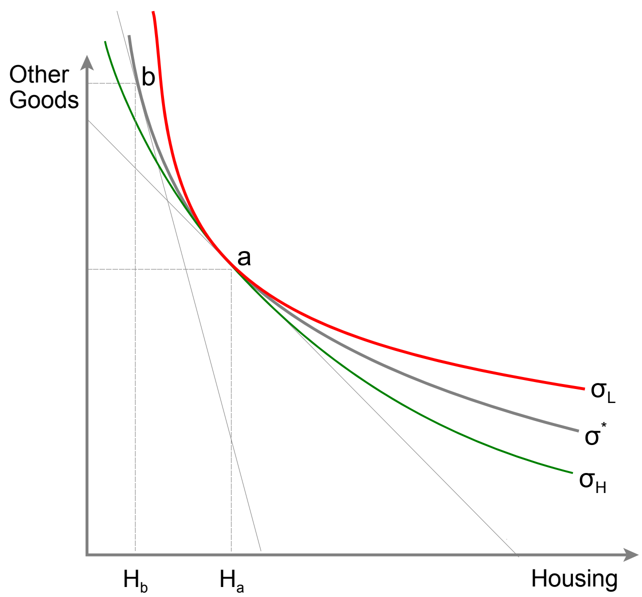
The derivation of demand functions which solve (2) involves solving two equations in two unknowns. The tangency condition which is a necessary condition for optimality equates the marginal rate of substitution and the price ratio:

$$\frac{\partial U/\partial H}{\partial U/\partial C} = \frac{(1-\alpha)H^{\rho-1}}{\alpha C^{\rho-1}} = p_H;$$

hence

$$\frac{H}{C} = \left(\frac{1-\alpha}{\alpha\ p_H}\right)^\sigma$$

Figure 1: Thomas's Choices



Substituting into the budget constraint, we have:

$$H = \frac{M}{p_H + \left(\frac{\alpha\ p_H}{1-\alpha}\right)^\sigma} = \frac{(1-\alpha)^\sigma M p_H^{-\sigma}}{\alpha^\sigma + (1-\alpha)^\sigma p_H^{1-\sigma}} \tag{3}$$

and

$$C = \frac{M}{1 + p_H \left(\frac{1-\alpha}{\alpha\ p_H}\right)^\sigma} = \frac{\alpha^\sigma M}{\alpha^\sigma + (1-\alpha)^\sigma p_H^{1-\sigma}} \tag{4}$$

3 Calibration

In order to evaluate Thomas's economic well-being in Zürich, we need to know the parameters of his utility function. It is conventional in applied general equilibrium analysis to employ exogenous elasticities and calibrated value values. If we follow this approach, σ is then exogenous and α is calibrated. We can calibrate on the basis of given values of benchmark value share of housing, $\theta = \bar{p}_H \bar{H}/\bar{M}$, choosing units so that the benchmark price of housing ($\bar{p}_H$) is unity. We then can substitute into equation (3) to obtain:

$$1 + \left(\frac{\alpha}{1-\alpha}\right)^\sigma = \frac{\bar{M}}{\bar{H}} = \frac{1}{\theta};$$

and hence

$$\alpha = \frac{(1-\theta)^{1/\sigma}}{\theta^{1/\sigma} + (1-\theta)^{1/\sigma}}. \tag{5}$$

4 Money Metric Utility

Substituting for α in $U(C,H)$, and denoting the base year expenditure on other goods as $\bar{C} = (1-\theta)\bar{M}$, we have

$$U(C,H) = \kappa \left((1-\theta)^{1/\sigma} C^\rho + \theta^{1/\sigma} H^\rho\right)^{1/\rho}$$

where the constant is define as:

$$\kappa = \left(\theta^{1/\sigma} + (1-\theta)^{1/\sigma}\right)^{-1/\rho}$$

The scale factor used to define utility, κ, may take on any positive value without altering the preference ordering. It is convenient to assign this value to the benchmark expenditure, so that utility can be measured in money-metric units at benchmark prices. Noting that $\theta^{1/\sigma} = \theta^{1-\rho}$, we then can write the utility function as:[1]

$$\tilde{U}(C,H) = \bar{M} \left((1-\theta)\left(\frac{C}{\bar{C}}\right)^\rho + \theta \left(\frac{H}{\bar{H}}\right)^\rho\right)^{1/\rho} \tag{6}$$

[1] $\tilde{U}$ can be related to *money metric utility*, $m(p,x)$, as defined in Varian (1992), Chapter 7. In Varian's notation, $m()$ is defined in terms of the expenditure function, $e()$ as:

$$m(p,x) = e(p,u(x)).$$

5 Indirect Utility

Substituting $C(p_H, M)$ and $H(p_H, M)$ into the utility function provides a convenient means of evaluating the welfare consequences of changes in prices or income level:

$$V(p_H, M) = U(C(p_H, M), M(p_H, M)) = \frac{M}{\left(\alpha^\sigma + (1-\alpha)^\sigma p_H^{1-\sigma}\right)^{1/(1-\sigma)}}$$

In money-metric terms, we have:[2]

$$\tilde{V}(p_H, M) = \frac{M}{(1 - \theta + \theta p_H^{1-\sigma})^{1/(1-\sigma)}}$$

We can then write demand functions in *calibrated share* form as:

$$H = \bar{H}\ \frac{\tilde{V}(p_H, M)}{\bar{M}} \left(\frac{p_U}{p_H}\right)^\sigma$$

$$C = \bar{C}\ \frac{\tilde{V}(p_H, M)}{\bar{M}} \left(\frac{p_U}{1}\right)^\sigma$$

where

$$p_U = \left(1 - \theta + \theta p_H^{1-\sigma}\right)^{1/(1-\sigma)}$$

p_U may be interpreted as the unit expenditure function, or an ideal price index.

These demand functions can be derived algebraically by substituting for α using 5 into 3 and 4, and noting that when the benchmark prices of both housing and other goods are unity, we have $\bar{H} = \theta\bar{M}$ and $\bar{C} = (1-\theta)\bar{M}$.

6 Should Thomas Move?

Thomas's welfare level in Zürich can be easily computed in money-metric terms as:

$$\tilde{V}(p_H = 3, M = 1.5) = \frac{1.5}{(0.7 + 0.3 \times 3^{1-\sigma})^{1/(1-\sigma)}}$$

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where $e(p, u)$ is the minimum expenditure required to achieve utility level u. Thus, money metric utility, $m(p, x)$, is the minimum income required at price level p to achieve the utility associated with consumption bundle x. In (6), we define the utility level of a bundle $x = \begin{pmatrix} C \\ H \end{pmatrix}$ as the minimum income required *at benchmark prices* to provide utility $U(x)$, i.e.

$$\tilde{U}(x) = m(\bar{p}, x),$$

in which $m(.)$ is Varian's definition of money metric utility.

[2] $\tilde{V}(p, M)$ can be intepreted as the income required *at benchmark prices* to achieve the utility level achievable at prices p and income M. This is related to Varian's money-metric utility and expenditure functions as:

$$\tilde{V}(p, M) = m(\bar{p}, x(p, M)) = e(\bar{p}, v(p, M))$$

in which $v(p, M)$ is the indirect (dual) utility function.

This expression cannot (to my knowledge) be solved in closed form, however it is easily to solve using Excel. The critical value for σ is that which equates welfare in Zürich with welfare level in Ann Arbor, i.e. $\tilde{V} = 1$. The numerical value is found to be $\sigma^* = 0.441$. The general dependence of welfare on the θ and σ can be illustrated in a contour diagram, as in Figure 2.

7 Generalization to Many Goods

In a general n-good model, a CES utility function can be described by n benchmark demands and the elasticity of substitution. If the benchmark consumption quantity for good i is c_i and the benchmark price is p_i, money-metric utility can be written as:

$$\tilde{U} = \bar{M}\left[\sum_i \theta_i \left(\frac{c_i}{\bar{c}_i}\right)^\rho\right]^{1/\rho}$$

where

$$\bar{M} = \sum_i \bar{p}_i \bar{c}_i$$

and

$$\theta_i = \frac{\bar{p}_i \bar{c}_i}{\bar{M}}$$

The indirect utility can be written as:

$$\tilde{V} = \frac{M}{p_U}$$

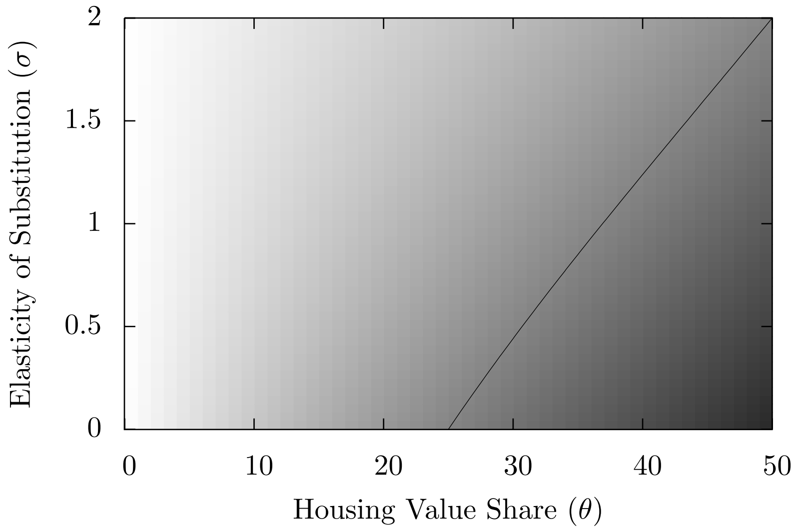
where

$$p_U = \left[\sum_i \theta_i \left(\frac{p_i}{\bar{p}_i}\right)^{1-\sigma}\right]^{1/(1-\sigma)}.$$

Demand functions then:

$$c_i(p, M) = \bar{c}_i \ \frac{\tilde{V}}{\bar{M}} \ \left(\frac{p_U \bar{p}_i}{p_i}\right)^\sigma.$$

Figure 2: Welfare Impacts of Price and Income Changes



The figure illustrates the welfare impact of a simultaneous 50% increase in income and 3-fold increase in the price of housing. The net impact depends on both the benchmark budget share of housing and the elasticity of substitution between housing and other goods. The line in this diagram indicates $\sigma^*(\theta)$. Shading indicates the welfare impact – darker areas are negative, lighter areas a positive. This figure was produced with the following GNUPLOT script:

```
set auto
set style data lines
set xlabel "Housing Value Share"
set ylabel "Elasticity of Substitution"
set view map
set contour base
set xrange [0:50]
set yrange [0:2]
set cntrparam levels discrete 0
set pm3d
set palette gray positive
unset title
unset key
unset colorbox
unset clabel
set isosamples 51,50; set samples 51,50
set xtics
set ytics
unset surface
splot 1.50/(1-x/100+x/100*3**(1-y))**(1/(1-y))-1
```

Bibliography

Varian, Hal, *Microeconomic Analysis*, W.W. Norton, 1992.